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Analytical solutions to Maxwell's equations in homogeneous media

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ANALYTICAL SOLUTIONS TO MAXWELL'S EQUATIONS IN HOMOGENEOUS MEDIA

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ABSTRACT

ANALYTICAL SOLUTIONS TO MAXWELL'S EQUATIONS IN HOMOGENEOUS MEDIA

by Carlos Urrutia

The coupled differential equations that dictate the behavior of electromagnetic phenomena are known collectively in the literature as Maxwell's equations. Analytical solutions to these equations are notoriously difficult to obtain and, in general, can only be obtained for problems possessing a great deal of symmetry. Numerical methods are used for solving problems involving more complicated geometries.

In this thesis, exact analytical solutions to Maxwell's equations are presented for the case of homogeneous media. The solutions are obtained using a novel mathematical approach. Detailed derivations are provided for a lossless homogeneous medium in one, two, and three dimensions. In addition, the newly obtained one-dimensional solutions are converted to discrete form apt for use in computer programs. Several examples of transmission lines and one-dimensional propagating waves using these discrete solutions are presented and the results discussed.

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Introduction

Unlike other fields of physics, the behavior of electromagnetic phenomena can be exactly described by an elegant set of equations. This remarkable result was first theorized in the nineteenth century by James Clerk Maxwell when he combined the work of his predecessors with his own, and formulated a unifying theory of electric and magnetic phenomena. Today, that theory is a set of partial differential equations that bears his name, and that countless experiments have confirmed over and over again.

In a medium containing only linear isotropic materials and *independent sources* of electric and magnetic energy, Maxwell's curl equations can be written as [1]

$$\frac{\partial \mathbf{D}(\mathbf{u},t)}{\partial t} = \nabla \times \mathbf{H}(\mathbf{u},t) - \mathbf{J}(\mathbf{u},t) \tag{1.1}$$

and

$$\frac{\partial \mathbf{B}(\mathbf{u},t)}{\partial t} = -\nabla \times \mathbf{E}(\mathbf{u},t), \tag{1.2}$$

[1] In these equations $\mathbf{u}$ is a physical location which, depending on the analysis, may have one (x), two (x, y), or three (x, y, z) coordinates.

where

- $\mathbf{D}(\mathbf{u},t)$ is the electric flux density (in Coulombs/meter^2),
- $\mathbf{E}(\mathbf{u},t)$ is the electric field (in Volts/meter),
- $\mathbf{B}(\mathbf{u},t)$ is the magnetic flux density (in Teslas),
- $\mathbf{H}(\mathbf{u},t)$ is the magnetic field (in Amperes/meter), and
- $\mathbf{J}(\mathbf{u},t)$ is the total electric current density (in Amperes/meter^2).

The electric and magnetic flux densities relate to the electric and magnetic field as

$$\mathbf{D}(\mathbf{u},t)=\varepsilon(\mathbf{u})\mathbf{E}(\mathbf{u},t) \tag{1.3}$$

and

$$\mathbf{B}(\mathbf{u},t)=\mu(\mathbf{u})\mathbf{H}(\mathbf{u},t), \tag{1.4}$$

where

- $\varepsilon(\mathbf{u})$ is the position-dependent electric permittivity (in Farads/meter), and
- $\mu(\mathbf{u})$ is the position-dependent magnetic permeability (in Henrys/meter).

As given, equations (1.1) and (1.2) are not symmetric, because only the former contains a current density term. To make the equations symmetric a magnetic current density term $\mathbf{M}(\mathbf{u},t)$ is introduced in equation (1.2). This inclusion, along with the relations given in (1.3) and (1.4), produces the modified equations

$$\frac{\partial \mathbf{E}(\mathbf{u},t)}{\partial t}=\frac{1}{\varepsilon(\mathbf{u})}\nabla\times\mathbf{H}(\mathbf{u},t)-\frac{1}{\varepsilon(\mathbf{u})}\mathbf{J}(\mathbf{u},t) \tag{1.5}$$

and

$$\frac{\partial \mathbf{H}(\mathbf{u},t)}{\partial t} = -\frac{1}{\mu(\mathbf{u})}\nabla\times\mathbf{E}(\mathbf{u},t) - \frac{1}{\mu(\mathbf{u})}\mathbf{M}(\mathbf{u},t), \tag{1.6}$$

where

- $\mathbf{M}(\mathbf{u},t)$ is the total magnetic current density (in Volts/meter^2).

The medium may contain materials that attenuate the $\mathbf{E}$ and $\mathbf{H}$ fields by absorbing or converting some of the field energy. In addition, there may be *independent sources* of electric and magnetic energy. The electric and magnetic current densities, $\mathbf{J}(\mathbf{u},t)$ and $\mathbf{M}(\mathbf{u},t)$, account for the presence of both types of elements as follows:

$$\mathbf{M}(\mathbf{u},t) = \sigma_m(\mathbf{u})\mathbf{H}(\mathbf{u},t) + \mathbf{M}_{src}(\mathbf{u},t) \tag{1.7}$$

and

$$\mathbf{J}(\mathbf{u},t) = \sigma_e(\mathbf{u})\mathbf{E}(\mathbf{u},t) + \mathbf{J}_{src}(\mathbf{u},t), \tag{1.8}$$

where

- $\sigma_e(\mathbf{u})$ is the position-dependent electric conductivity (in Siemens/meter),
- $\sigma_m(\mathbf{u})$ is the position-dependent equivalent magnetic loss (in Ohms/meter),
- $\mathbf{J}_{src}(\mathbf{u},t)$ is the total contribution of *independent sources* of electric current density (in Amperes/meter^2), and
- $\mathbf{M}_{src}(\mathbf{u},t)$ is the total contribution of *independent sources* of magnetic current density (in Volts/meter^2).

Although it is possible to obtain exact solutions to equations (1.1) to (1.4) for some particular cases by exploiting symmetry, in general, one must resort to numerical methods to solve them. Existing numerical techniques are mainly of two types: time-domain and frequency domain. Time-domain methods offer great flexibility for material inhomogeneities and allow the user to visualize the propagation and scattering of the fields as time progresses. Frequency-domain methods are very useful when a spectrum needs to be computed, or when periodicity is present in the structure being modeled.

In this document a novel method for directly solving Maxwell's equations in the time-domain is presented. Using this method, analytical solutions for a lossless homogeneous medium are derived in one, two, and three dimensions. Subsequently, the one-dimensional solution obtained in the analysis is converted to a discrete form that can be used in computer simulations. Examples of one-dimensional wave propagation and transmission lines are presented. Finally, a method for obtaining discrete versions of the two- and three-dimensional solutions using the sampling theorem is discussed.

1.1 Overview

The outline of this thesis is as follows:

- **Chapter 2.** In this chapter, a novel method for obtaining analytical solutions to Maxwell's equations is introduced. Detailed derivations of the solutions in one, two, and three dimensions are presented and discussed.
- **Chapter 3** In this chapter, the one-dimensional solution obtained in Chapter 2 is converted from continuous to discrete. The discrete solution is implemented in

MATLAB, several examples are simulated, and the results presented. A method for producing numerical solutions in two and three dimensions is then discussed.

- **Chapter 4** In this chapter, aspects of the theoretical work that were not covered in this thesis, but may be worth pursuing in future research, are discussed.

Chapter 2

Analytical Solutions to Maxwell's Equations

In this chapter, exact analytical solutions to Maxwell's equations in a homogeneous medium are obtained. The key steps of the approach are the transformation of Maxwell's partial differential equations from real- to inverse-space into a system of coupled ordinary differential equations (ODEs), and the removal of the differential operators of the curls by means of the Fourier transform. The solution of the resulting system of coupled ODEs can be readily found.

In the first chapter of this document, Maxwell's curl equations for a medium containing only isotropic materials and *independent sources* of electric and magnetic energy were introduced. The equations are

$$\frac{\partial \mathbf{E}(\mathbf{u},t)}{\partial t}=\frac{1}{\varepsilon(\mathbf{u})}\nabla\times\mathbf{H}(\mathbf{u},t)-\frac{1}{\varepsilon(\mathbf{u})}\mathbf{J}(\mathbf{u},t) \tag{2.1}$$

and

$$\frac{\partial \mathbf{H}(\mathbf{u},t)}{\partial t}=-\frac{1}{\mu(\mathbf{u})}\nabla\times\mathbf{E}(\mathbf{u},t)-\frac{1}{\mu(\mathbf{u})}\mathbf{M}(\mathbf{u},t), \tag{2.2}$$

where

$$\mathbf{M}(\mathbf{u},t) = \sigma_m(\mathbf{u})\mathbf{H}(\mathbf{u},t) + \mathbf{M}_{src}(\mathbf{u},t), \tag{2.3}$$

and

$$\mathbf{J}(\mathbf{u},t) = \sigma_e(\mathbf{u})\mathbf{E}(\mathbf{u},t) + \mathbf{J}_{src}(\mathbf{u},t). \tag{2.4}$$

The intent is to transform these partial differential equations into a system of coupled ordinary differential equations. To this end, equations (2.1) and (2.2) are written in matrix form as

$$\frac{\partial}{\partial t}\mathbf{g}(\mathbf{u},t) = \hat{\mathbf{P}}(\mathbf{u})\mathbf{g}(\mathbf{u},t) + \mathbf{r}(\mathbf{u},t) \tag{2.5}$$

and

$$\mathbf{g}(\mathbf{u},t_0) = \mathbf{g}(\mathbf{u}_0), \tag{2.6}$$

where $\mathbf{u} = (x,y,z)$ is the direction vector and $\mathbf{g}(\mathbf{u}_0)$ represents the initial conditions at time t_0. The operator matrix $\hat{\mathbf{P}}(\mathbf{u})$ and the vectors $\mathbf{g}(\mathbf{u},t)$ and $\mathbf{r}(\mathbf{u},t)$ are given by

$$\mathbf{g}(\mathbf{u},t) = \begin{bmatrix} \mathbf{E}(\mathbf{u},t) \\ \mathbf{H}(\mathbf{u},t) \end{bmatrix}, \tag{2.7}$$

$$\hat{\mathbf{P}}(\mathbf{u}) = \begin{bmatrix} -\dfrac{\sigma_e(\mathbf{u})}{\varepsilon(\mathbf{u})} & \dfrac{1}{\varepsilon(\mathbf{u})}\nabla\times \\ -\dfrac{1}{\mu(\mathbf{u})}\nabla\times & -\dfrac{\sigma_m(\mathbf{u})}{\mu(\mathbf{u})} \end{bmatrix}, \tag{2.8}$$

and

$$\mathbf{r}(\mathbf{u},t) = \begin{bmatrix} -\dfrac{1}{\varepsilon(\mathbf{u})}\mathbf{J}_{src}(\mathbf{u},t) \\ -\dfrac{1}{\mu(\mathbf{u})}\mathbf{M}_{src}(\mathbf{u},t) \end{bmatrix}. \tag{2.9}$$

It is important to point out that, for the sake of generality, equations (2.5) to (2.9) are written using symbolic notation. The vector $\mathbf{u}$ can be one-, two-, or three-dimensional and the curls in equation (2.8) produce a $\hat{\mathbf{P}}(\mathbf{u})$ matrix of size 2x2, 3x3, or 6x6 depending on the dimensionality. The detailed equations are shown explicitly for each case in later sections of this chapter.

The kernel matrix $\hat{\mathbf{P}}(\mathbf{u})$ contains the differential operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ from the curl operation. Attempts could be made to approximate this matrix and then solve the equations numerically, but since the goal is obtaining analytic solutions, there is a need to be rid of the differential operators. To accomplish this, equation (2.5) is mapped from real- to inverse-space (also called k-space or reciprocal space) using the Fourier transform (appendix A) and its properties. Applying the Fourier transform to both sides of equations (2.5) yields

$$\frac{\partial}{\partial t}\int_{\mathbf{u}}\mathbf{g}(\mathbf{u},t)e^{-i2\pi\mathbf{k}\cdot\mathbf{u}}d\mathbf{u}=\int_{\mathbf{u}}\hat{\mathbf{P}}(\mathbf{u})\mathbf{g}(\mathbf{u},t)e^{-i2\pi\mathbf{k}\cdot\mathbf{u}}d\mathbf{u}+\int_{\mathbf{u}}\mathbf{r}(\mathbf{u},t)e^{-i2\pi\mathbf{k}\cdot\mathbf{u}}d\mathbf{u}, \tag{2.10}$$

where $\mathbf{k}=(p,q,r)$ is the direction vector in inverse space.

The product $\hat{\mathbf{P}}(\mathbf{u})e^{-i2\pi\mathbf{k}\cdot\mathbf{u}}$ transforms the differential operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ to $-i2\pi pe^{-i2\pi\mathbf{k}\cdot\mathbf{u}}$, $-i2\pi qe^{-i2\pi\mathbf{k}\cdot\mathbf{u}}$, and $-i2\pi re^{-i2\pi\mathbf{k}\cdot\mathbf{u}}$, respectively. Using the convolution property of the Fourier transform (appendix A) equation (2.10) can be written as

$$\frac{\partial}{\partial t}\mathbf{G}(\mathbf{k},t)=\int_{\hat{\mathbf{k}}}\mathbf{P}(\hat{\mathbf{k}})\mathbf{G}(\mathbf{k}-\hat{\mathbf{k}},t)d\hat{\mathbf{k}}+\mathbf{R}(\mathbf{k},t), \tag{2.11}$$

where the terms $\mathbf{P}(\mathbf{k})$, $\mathbf{G}(\mathbf{k},t)$, and $\mathbf{R}(\mathbf{k},t)$ are the Fourier transforms of $\hat{\mathbf{P}}(\mathbf{u})$, $\mathbf{g}(\mathbf{u},t)$, and $\mathbf{r}(\mathbf{u},t)$ respectively.

Up until this moment no assumptions have been made about the nature of the medium. Equation (2.11) contains all the information about the different regions that may be present in the medium. Unfortunately, preserving this generality makes the search for an analytical solution extremely difficult, if not impossible. An analytical solution is possible, however, if the medium is homogeneous. In this case, the elements of matrix $\hat{\mathbf{P}}(\mathbf{u})$ are constants, that is $\varepsilon(\mathbf{u})=\varepsilon$, $\mu(\mathbf{u})=\mu$, $\sigma_e(\mathbf{u})=\sigma_e$, and $\sigma_m(\mathbf{u})=\sigma_m$. When the Fourier transform is applied, the constants in real-space are transformed into delta symbols in inverse-space, and equation (2.11) becomes

$$\frac{\partial}{\partial t}\mathbf{G}(\mathbf{k},t)=\mathbf{P}(\mathbf{k})\mathbf{G}(\mathbf{k},t)+\mathbf{R}(\mathbf{k},t). \tag{2.12}$$

This is a regular system of coupled ordinary differential equations. The solution to this kind of system can be found in any good textbook that discusses systems of ordinary differential equations (i.e. refer to [1]). For the case of equation (2.12) the solution is given by

$$\mathbf{G}(\mathbf{k},t)=\mathbf{H}(\mathbf{k},t-t_0)\mathbf{G}(\mathbf{k},t_0)+\int_{t_0}^{t}\mathbf{H}(\mathbf{k},t-\tau)\mathbf{R}(\mathbf{k},\tau)d\tau, \tag{2.13}$$

where the term $\mathbf{H}(\mathbf{k},t)$ is an exponential matrix (see appendix B) and can be written as

$$\mathbf{H}(\mathbf{k},t)=e^{\mathbf{P}(\mathbf{k})t}=\mathbf{M}(\mathbf{k})\mathbf{D}\left(e^{\lambda_n t}\right)\mathbf{M}^{-1}(\mathbf{k}), \tag{2.14}$$

where $\mathbf{M}(\mathbf{k})$ is a modal matrix whose columns are the eigenvectors of $\mathbf{P}(\mathbf{k})$ and $\mathbf{D}\left(e^{\lambda_n t}\right)$ is a diagonal matrix whose elements are the exponentials $e^{\lambda_n t}$ where the λ_n's are the eigenvalues of $\mathbf{P}(\mathbf{k})$.

Applying the inverse Fourier transform to equation (2.13) and using the convolution property yields

$$\begin{aligned}\mathbf{g}(\mathbf{u},t)=&\left(\int_{\mathbf{k}}\mathbf{H}(\mathbf{k},t-t_0)e^{i2\pi\mathbf{k}\cdot\mathbf{u}}d\mathbf{k}\right)*\mathbf{g}(\mathbf{u},t_0)\\&+\int_{t_0}^{t}\left(\int_{\mathbf{k}}\mathbf{H}(\mathbf{k},t-\tau)\left(\int_{\hat{\mathbf{u}}}\mathbf{r}(\hat{\mathbf{u}},\tau)e^{-i2\pi\mathbf{k}\cdot\hat{\mathbf{u}}}d\hat{\mathbf{u}}\right)e^{i2\pi\mathbf{k}\cdot\mathbf{u}}d\mathbf{k}\right)d\tau,\end{aligned} \tag{2.15}$$

where $\mathbf{R}(\mathbf{k},\tau)$ has been explicitly written as the Fourier transform of $\mathbf{r}(\mathbf{u},\tau)$. Interchanging the order of the integrals[1] on the second term on the right-hand side yields

$$\begin{aligned}\mathbf{g}(\mathbf{u},t)=&\int_{\hat{\mathbf{u}}}\left(\int_{\mathbf{k}}\mathbf{H}(\mathbf{k},t-t_0)e^{i2\pi\mathbf{k}\cdot(\mathbf{u}-\hat{\mathbf{u}})}d\mathbf{k}\right)\mathbf{g}(\hat{\mathbf{u}},t_0)\,d\hat{\mathbf{u}}\\&+\int_{t_0}^{t}\left(\int_{\hat{\mathbf{u}}}\left(\int_{\mathbf{k}}\mathbf{H}(\mathbf{k},t-\tau)e^{i2\pi\mathbf{k}\cdot(\mathbf{u}-\hat{\mathbf{u}})}d\mathbf{k}\right)\mathbf{r}(\hat{\mathbf{u}},\tau)\,d\hat{\mathbf{u}}\right)d\tau.\end{aligned} \tag{2.16}$$

Therefore, the real-space solution to equation (2.5) for a *homogeneous* medium is

$$\mathbf{g}(\mathbf{u},t)=\int_{\hat{\mathbf{u}}}\mathbf{h}(\mathbf{u}-\hat{\mathbf{u}},t-t_0)\mathbf{g}(\hat{\mathbf{u}},t_0)\,d\hat{\mathbf{u}}+\int_{t_0}^{t}\int_{\hat{\mathbf{u}}}\mathbf{h}(\mathbf{u}-\hat{\mathbf{u}},t-\tau)\mathbf{r}(\hat{\mathbf{u}},\tau)\,d\hat{\mathbf{u}}\,d\tau, \tag{2.17}$$

[1] The function must converge to the same value in both cases for this to be valid.

where $\mathbf{h}(\mathbf{u},t)$ and $\mathbf{r}(\mathbf{u},t)$ are

$$\mathbf{h}(\mathbf{u},t) = \int_{\mathbf{k}} \mathbf{H}(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{u}} d\mathbf{k} = \int_{\mathbf{k}} e^{\mathbf{P}(\mathbf{k})t} e^{i\mathbf{k}\cdot\mathbf{u}} d\mathbf{k}, \tag{2.18}$$

and

$$\mathbf{r}(\mathbf{u},t) = \begin{bmatrix} -\frac{1}{\varepsilon}\mathbf{J}_{source}(\mathbf{u},t) \\ -\frac{1}{\mu}\mathbf{M}_{source}(\mathbf{u},t) \end{bmatrix}. \tag{2.19}$$

The meaning of the solution given by equation (2.17) is clear. Given an initial time t_0 and a final time $t > t_0$, all that is needed to determine the fields at time t are the kernel matrix $\mathbf{h}(\mathbf{u},t)$, the fields $\mathbf{g}(\mathbf{u},t)$ at time t_0, and any contributions from independent sources $\mathbf{r}(\mathbf{u},t)$ from the initial time t_0 to the final time t.

In the sections that follow, we study particular cases of equation (2.5) and find the kernel function $\mathbf{h}(\mathbf{u},t)$ of equation (2.17) for the one-, two-, and three-dimensional cases.

2.1 One-dimensional Equation

In the preceding section the solution to equation (2.5) was obtained for a homogeneous medium. The formulas obtained therein, however, are of a general nature as they apply to one-, two-, and three-dimensional spaces alike. In this section the same steps are taken but the focus is the one-dimensional equation. For this case an x-directed, y-polarized one dimensional plane wave with components $E_y(x,t)$ and $H_z(x,t)$ is considered. In this case, equation (2.5) is given by

$$\frac{\partial}{\partial t}\mathbf{g}(x,t)=\begin{bmatrix} -\frac{\sigma_e(x)}{\varepsilon(x)} & \frac{-1}{\varepsilon(x)}\frac{\partial}{\partial x} \\ \frac{-1}{\mu(x)}\frac{\partial}{\partial x} & -\frac{\sigma_m(x)}{\mu(x)} \end{bmatrix}\mathbf{g}(x,t)+\mathbf{r}(x,t), \tag{2.20}$$

where

$$\mathbf{g}(x,t)=\begin{bmatrix} E_y(x,t) \\ H_z(x,t) \end{bmatrix} \tag{2.21}$$

and

$$\mathbf{r}(x,t)=\begin{bmatrix} -\frac{1}{\varepsilon(x)} & 0 \\ 0 & -\frac{1}{\mu(x)} \end{bmatrix}\begin{bmatrix} \mathrm{J}_{src}(x,t) \\ \mathrm{M}_{src}(x,t) \end{bmatrix}. \tag{2.22}$$

Proceeding in the same manner as section one, our next task is to transform equation (2.20) from real-space (x-domain) to inverse-space (p-domain). Taking the Fourier transform on both sides of equation (2.20) yields

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\begin{bmatrix}E_y(x,t)\\H_z(x,t)\end{bmatrix}e^{-i2\pi px}dx=\int_{-\infty}^{\infty}\begin{bmatrix}-\dfrac{\sigma_e(x)}{\varepsilon(x)} & \dfrac{-1}{\varepsilon(x)}\dfrac{\partial}{\partial x}\\ \dfrac{-1}{\mu(x)}\dfrac{\partial}{\partial x} & -\dfrac{\sigma_m(x)}{\mu(x)}\end{bmatrix}\begin{bmatrix}E_y(x,t)\\H_z(x,t)\end{bmatrix}e^{-i2\pi px}dx$$
$$+\int_{-\infty}^{\infty}\begin{bmatrix}-\dfrac{1}{\varepsilon(x)} & 0\\ 0 & -\dfrac{1}{\mu(x)}\end{bmatrix}\begin{bmatrix}J_{src}(x,t)\\M_{src}(x,t)\end{bmatrix}e^{-i2\pi px}dx. \tag{2.23}$$

Moving the differential operator $\partial/\partial t$ outside of the integral and making the substitutions $F_1(x)=-\sigma_e(x)/\varepsilon(x)$, $F_2(x)=-1/\varepsilon(x)$, $F_3(x)=-1/\mu(x)$, and $F_4(x)=-\sigma_m(x)/\mu(x)$ gives

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\begin{bmatrix}E_y(x,t)\\H_z(x,t)\end{bmatrix}e^{-i2\pi px}dx=\int_{-\infty}^{\infty}\begin{bmatrix}F_1(x) & F_2(x)\dfrac{\partial}{\partial x}\\ F_3(x)\dfrac{\partial}{\partial x} & F_4(x)\end{bmatrix}\begin{bmatrix}E_y(x,t)\\H_z(x,t)\end{bmatrix}e^{-i2\pi px}dx$$
$$+\int_{-\infty}^{\infty}\begin{bmatrix}F_2(x) & 0\\ 0 & F_3(x)\end{bmatrix}\begin{bmatrix}J_{src}(x,t)\\M_{src}(x,t)\end{bmatrix}e^{-i2\pi px}dx. \tag{2.24}$$

To complete the transformation of equation (2.24) to inverse space, we use the following properties of the Fourier transform (see appendix A):

$$\int_{-\infty}^{\infty}m(x)n(x)e^{-i2\pi px}dx=M(p)*N(p)=\int_{-\infty}^{\infty}M(p-\hat{p})N(\hat{p})d\hat{p} \tag{2.25}$$

and

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}m(x)\right\}e^{-i2\pi px}dx=i2\pi pM(p). \tag{2.26}$$

The resulting equation is

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} E_y(p,t) \\ H_z(p,t) \end{bmatrix} &= \int_{-\infty}^{\infty}\begin{bmatrix} F_1(p-\hat{p}) & i2\pi\hat{p}F_2(p-\hat{p}) \\ i2\pi\hat{p}F_3(p-\hat{p}) & F_4(p-\hat{p}) \end{bmatrix}\begin{bmatrix} E_y(\hat{p},t) \\ H_z(\hat{p},t) \end{bmatrix} d\hat{p} \\
&+ \int_{-\infty}^{\infty}\begin{bmatrix} F_2(p-\hat{p}) & 0 \\ 0 & F_3(p-\hat{p}) \end{bmatrix}\begin{bmatrix} J_{src}(\hat{p},t) \\ M_{src}(\hat{p},t) \end{bmatrix} d\hat{p}.
\end{aligned} \tag{2.27}
$$

Equation (2.27) is the one-dimensional version of equation (2.11): an integro-differential equation that contains all the information about the medium, but whose analytical solution may be difficult to obtain, if obtainable at all. In the sections that follow a solution to this equation is obtained for the special case of a homogeneous medium.

2.1.1 Solution to the One-Dimensional Equation in a Homogeneous Medium

In the special case of a *homogeneous* medium, where ε, μ, σ, and ρ are constants, the terms $F_1(x)$, $F_2(x)$, $F_3(x)$, and $F_4(x)$ become $F_1(p)=(-\sigma_e/\varepsilon)\delta(p)$, $F_2(p)=(-1/\varepsilon)\delta(p)$, $F_3(p)=(-1/\mu)\delta(p)$, and $F_4(x)=(-\sigma_m/\mu)\delta(p)$ in inverse space. In this case the delta symbols eliminate the integral and equation (2.27) becomes

$$
\frac{\partial \mathbf{G}(p,t)}{\partial t} = \mathbf{P}(p)\mathbf{G}(p,t) + \mathbf{R}(p,t), \tag{2.28}
$$

where

$$\mathbf{P}(p)=\begin{bmatrix} -\frac{\sigma_e}{\varepsilon} & -\frac{i2\pi p}{\varepsilon} \\ -\frac{i2\pi p}{\mu} & -\frac{\sigma_m}{\mu} \end{bmatrix}, \tag{2.29}$$

$$\mathbf{G}(p,t)=\begin{bmatrix} E_y(p,t) \\ H_z(p,t) \end{bmatrix}, \tag{2.30}$$

and

$$\mathbf{R}(p,t)=\begin{bmatrix} -\frac{1}{\varepsilon}J_{src}(p,t) \\ -\frac{1}{\mu}M_{src}(p,t) \end{bmatrix}. \tag{2.31}$$

The solution to equation (2.28) at time t is the one-dimensional version of equation (2.13) given by

$$\mathbf{G}(p,t)=\mathbf{H}(p,t-t_0)\mathbf{G}(p,t_0)+\int_{t_0}^{t}\mathbf{H}(p,t-\tau)\mathbf{R}(p,\tau)d\tau\,, \tag{2.32}$$

where $\mathbf{G}(p,t_0)$ is the solution at time t_0.

In the special case of a one-dimensional lossless homogeneous medium, where $\sigma_e=0$, $\sigma_m=0$, and ε and μ are constants, equation (2.29) is

$$\mathbf{P}(p)=\begin{bmatrix} 0 & -\frac{i2\pi p}{\varepsilon} \\ -\frac{i2\pi p}{\mu} & 0 \end{bmatrix}. \tag{2.33}$$

The kernel matrix $\mathbf{H}(p,t)$ in this case (see appendix B) is given by

$$\mathbf{H}(p,t)=e^{\mathbf{P}(p)t}=\begin{bmatrix}\cos(2\pi pvt) & -i\eta\sin(2\pi pvt)\\ -\frac{i}{\eta}\sin(2\pi pvt) & \cos(2\pi pvt)\end{bmatrix}, \tag{2.34}$$

where $v=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.

Applying the inverse Fourier transform to equation (2.32) yields

$$\mathbf{g}(x,t)=\int_{-\infty}^{\infty}\mathbf{h}(x-\hat{x},t-t_0)\mathbf{g}(\hat{x},t_0)\,d\hat{x}+\int_{t_0}^{t}\int_{-\infty}^{\infty}\mathbf{h}(x-\hat{x},t-\tau)\mathbf{r}(\hat{x},\tau)\,d\hat{x}\,d\tau. \tag{2.35}$$

This is the one-dimensional form of equation (2.17). The forcing function $\mathbf{r}(x,t)$, initial conditions $\mathbf{g}(x,t_0)$, and kernel matrix $\mathbf{h}(x,t)$ are given by

$$\mathbf{r}(x,t)=\begin{bmatrix}-\frac{1}{\varepsilon}J_{src}(x,t)\\ -\frac{1}{\mu}M_{src}(x,t)\end{bmatrix}, \tag{2.36}$$

$$\mathbf{g}(x,t_0)=\begin{bmatrix}E_y(x,t_0)\\ H_z(x,t_0)\end{bmatrix}, \tag{2.37}$$

and

$$\begin{aligned}\mathbf{h}(x,t)&=\int_{-\infty}^{\infty}\begin{bmatrix}\cos(2\pi pvt) & -i\eta\sin(2\pi pvt)\\ -\frac{i}{\eta}\sin(2\pi pvt) & \cos(2\pi pvt)\end{bmatrix}e^{i2\pi px}dp\\ &=\begin{bmatrix}\frac{1}{2}\left[\delta(x+vt)+\delta(x-vt)\right] & -\frac{\eta}{2}\left[\delta(x+vt)-\delta(x-vt)\right]\\ -\frac{1}{2\eta}\left[\delta(x+vt)-\delta(x-vt)\right] & \frac{1}{2}\left[\delta(x+vt)+\delta(x-vt)\right]\end{bmatrix}.\end{aligned} \tag{2.38}$$

Substituting the kernel matrix $\mathbf{h}(x,t)$ into equation (2.38) yields

$$
\begin{aligned}
E_y(x,t) = & \frac{1}{2}\left[E_y\left(x+v(t-t_0),t_0\right)+E_y\left(x-v(t-t_0),t_0\right)\right] \\
- & \frac{\eta}{2}\left[H_z\left(x+v(t-t_0),t_0\right)-H_z\left(x-v(t-t_0),t_0\right)\right] \\
- & \int_{t_0}^{t}\frac{1}{2\varepsilon}\left[J_0\left(x+v(t-\tau),\tau\right)+J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\
+ & \int_{t_0}^{t}\frac{\eta}{2\mu}\left[M_0\left(x+v(t-\tau),\tau\right)-M_0\left(x-v(t-\tau),\tau\right)\right]d\tau
\end{aligned}
\tag{2.39}
$$

and

$$
\begin{aligned}
H_z(x,t) = & -\frac{1}{2\eta}\left[E_y\left(x+v(t-t_0),t_0\right)-E_y\left(x-v(t-t_0),t_0\right)\right] \\
+ & \frac{1}{2}\left[H_z\left(x+v(t-t_0),t_0\right)+H_z\left(x-v(t-t_0),t_0\right)\right] \\
+ & \int_{t_0}^{t}\frac{1}{2\eta\varepsilon}\left[J_0\left(x+v(t-\tau),\tau\right)-J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\
- & \int_{t_0}^{t}\frac{1}{2\mu}\left[M_0\left(x+v(t-\tau),\tau\right)+M_0\left(x-v(t-\tau),\tau\right)\right]d\tau.
\end{aligned}
\tag{2.40}
$$

Expressions (2.39) and (2.40) are exact analytical solutions for the one-dimensional form of Maxwell's equations in a lossless homogeneous medium. Given times t_0 and t, where $t > t_0$, if the electric and magnetic fields are known at t_0 and the current and magnetic sources are known from t_0 to t, these equations can be used to compute the electric and magnetic fields at time t.

2.1.2 Interpretation of the One-Dimensional Solution in a Homogeneous Medium

The one-dimensional solution in a lossless homogeneous medium has a special interpretation. In this case, it is possible to write the solution as the sum of two traveling waves: one moving forward and the other backwards. By making the substitution $t = t_0 + \phi$, we can rewrite equations (2.39) and (2.40) as

$$
\begin{aligned}
E_y(x,t_0+\phi) = &\left\{\frac{1}{2}E_y(x - v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{1}{2\varepsilon}J_0\left(x - v(t_0+\phi-\tau),\tau\right)d\tau \right.\\
&\left. + \frac{\eta}{2}H_z(x - v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{\eta}{2\mu}M_0\left(x - v(t_0+\phi-\tau),\tau\right)d\tau\right\}\\
&+\left\{\frac{1}{2}E_y(x + v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{1}{2\varepsilon}J_0\left(x + v(t_0+\phi-\tau),\tau\right)d\tau \right.\\
&\left. - \frac{\eta}{2}H_z(x + v\phi, t_0) + \int_{t_0}^{t_0+\phi}\frac{\eta}{2\mu}M_0\left(x + v(t_0+\phi-\tau),\tau\right)d\tau\right\}
\end{aligned}
\tag{2.41}
$$

and

$$
\begin{aligned}
H_z(x,t_0+\phi) = &\left\{\frac{1}{2\eta}E_y(x - v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{1}{2\eta\varepsilon}J_0\left(x - v(t_0+\phi-\tau),\tau\right)d\tau \right.\\
&\left. + \frac{1}{2}H_z(x - v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{1}{2\mu}M_0\left(x - v(t_0+\phi-\tau),\tau\right)d\tau\right\}\\
&+\left\{-\frac{1}{2\eta}E_y(x + v\phi, t_0) + \int_{t_0}^{t_0+\phi}\frac{1}{2\eta\varepsilon}J_0\left(x + v(t_0+\phi-\tau),\tau\right)d\tau \right.\\
&\left. + \frac{1}{2}H_z(x + v\phi, t_0) - \int_{t_0}^{t_0+\phi}\frac{1}{2\mu}M_0\left(x + v(t_0+\phi-\tau),\tau\right)d\tau\right\}.
\end{aligned}
\tag{2.42}
$$

If we further restrict the medium to be sourceless, the equations become

$$E_y(x,t_0+\phi)=\left\{\frac{1}{2}E_y(x-v\phi,t_0)+\frac{\eta}{2}H_z(x-v\phi,t_0)\right\}$$
$$+\left\{\frac{1}{2}E_y(x+v\phi,t_0)-\frac{\eta}{2}H_z(x+v\phi,t_0)\right\} \tag{2.43}$$

and

$$H_z(x,t_0+\phi)=\left\{\frac{1}{2\eta}E_y(x-v\phi,t_0)+\frac{1}{2}H_z(x-v\phi,t_0)\right\}$$
$$+\left\{-\frac{1}{2\eta}E_y(x+v\phi,t_0)+\frac{1}{2}H_z(x+v\phi,t_0)\right\}. \tag{2.44}$$

The first bracketed term on the right hand side of equations (2.43) and (2.44) corresponds to the forward-traveling wave, and the second one to the backward-traveling wave. Since the medium is homogeneous (i.e. the parameters are constant), waves travel at the same speed everywhere. Therefore, to determine the strength of the field at location x_0, ϕ time units later, one only needs to look at the strength of the waves at locations $x_0+v\phi$ and $x_0-v\phi$. This is because the waves travel with speed v and will arrive at location x_0 exactly ϕ time units later

2.1.3 Time Evolution of One-Dimensional Plane Waves

In this section it is shown that in a source-free lossless homogeneous medium, application of the kernel matrix $\mathbf{H}(p,t-t_0)$ given in (2.34) to a plane wave at time t_0 results in the plane wave at time t. For a homogeneous medium with $(\mathbf{J}_{source}=\mathbf{M}_{source}=0)$ and $\sigma_e=\sigma_m=0$, the system of one-dimensional Maxwell's curl equations in matrix form is

$$\frac{\partial}{\partial t}\begin{bmatrix} E_y(x,t) \\ H_z(x,t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial x} \\ -\frac{1}{\mu}\frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} E_y(x,t) \\ H_z(x,t) \end{bmatrix}. \tag{2.45}$$

An x-directed, y-polarized one-dimensional plane wave with wavenumber $\hat{p}$ and components $E_y(x,t)$ and $H_z(x,t)$ is defined as

$$\begin{bmatrix} E_y(x,t) \\ H_z(x,t) \end{bmatrix} = \begin{bmatrix} E_y^+ \\ H_z^+ \end{bmatrix} e^{i2\pi\hat{p}(x-vt)} + \begin{bmatrix} E_y^- \\ H_z^- \end{bmatrix} e^{i2\pi\hat{p}(x+vt)}. \tag{2.46}$$

Substituting (2.46) into equations (2.45) produces

$$\begin{aligned} &-i2\pi\hat{p}vE_y^+ e^{i2\pi\hat{p}(x-vt)} + i2\pi\hat{p}vE_y^- e^{i2\pi\hat{p}(x+vt)} \\ &\quad = -\frac{1}{\varepsilon}\left[i2\pi\hat{p}H_z^+ e^{i2\pi\hat{p}(x-vt)} + i2\pi\hat{p}H_z^- e^{i2\pi\hat{p}(x+vt)}\right] \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} &-i2\pi\hat{p}vH_z^+ e^{i2\pi\hat{p}(x-vt)} + i2\pi\hat{p}vH_z^- e^{i2\pi\hat{p}(x+vt)} \\ &\quad = -\frac{1}{\mu}\left[i2\pi\hat{p}E_y^+ e^{i2\pi\hat{p}(x-vt)} + i2\pi\hat{p}E_y^- e^{i2\pi\hat{p}(x+vt)}\right]. \end{aligned} \tag{2.48}$$

Equating the common terms yields

$$H_z^+ = \varepsilon v E_y^+ \qquad\qquad H_z^- = -\varepsilon v E_y^- \tag{2.49}$$

and

$$H_z^+ = \frac{1}{\mu v}E_y^+ \qquad\qquad H_z^- = -\frac{1}{\mu v}E_y^-. \tag{2.50}$$

As expected, to have a solvable system it is required that $v = 1/\sqrt{\mu\varepsilon}$. Substitution of v into equations (2.49) and (2.50) produces

$$H_z^+ = \frac{1}{\eta} E_y^+ \qquad \text{and} \qquad H_z^- = -\frac{1}{\eta} E_y^-, \tag{2.51}$$

where $\eta = \sqrt{\mu/\varepsilon}$. These identities are used later in the proof.

The first task is to transform the plane wave from real- to inverse-space. Applying the Fourier transform to equation (2.46) and evaluating at time t_0 yields

$$\begin{bmatrix} E_y(p,t) \\ H_z(p,t) \end{bmatrix} = \left(\begin{bmatrix} E_y^+ \\ H_z^+ \end{bmatrix} e^{-i2\pi\hat{p}vt} + \begin{bmatrix} E_y^- \\ H_z^- \end{bmatrix} e^{i2\pi\hat{p}vt} \right) \delta(p-\hat{p}). \tag{2.52}$$

The next step is the application of the kernel matrix $\mathbf{H}(p,t-t_0)$ to the inverse-space plane wave of equation (2.52). If the derivations thus far have been correct, this operation should produce the same plane wave at time t. Multiplying the kernel matrix $\mathbf{H}(p,t-t_0)$ with the plane wave produces

$$\begin{bmatrix} P_1(p,t,t_0) \\ P_2(p,t,t_0) \end{bmatrix} = \begin{bmatrix} \cos\left(2\pi pv(t-t_0)\right) & -i\eta\sin\left(2\pi pv(t-t_0)\right) \\ -\frac{i}{\eta}\sin\left(2\pi pv(t-t_0)\right) & \cos\left(2\pi pv(t-t_0)\right) \end{bmatrix} \begin{bmatrix} E_y(p,t_0) \\ H_z(p,t_0) \end{bmatrix}. \tag{2.53}$$

Substituting (2.52) into equation (2.53) yields

$$\begin{aligned} P_1(p,t,t_0) &= \left(\frac{e^{i2\pi pv(t-t_0)} + e^{-i2\pi pv(t-t_0)}}{2} \right) \left[E_y^+ e^{-i2\pi\hat{p}vt_0} + E_y^- e^{i2\pi\hat{p}vt_0} \right] \delta(p-\hat{p}) \\ &- i\eta \left(\frac{e^{i2\pi pv(t-t_0)} - e^{-i2\pi pv(t-t_0)}}{2i} \right) \left[H_z^+ e^{-i2\pi\hat{p}vt_0} + H_z^- e^{i2\pi\hat{p}vt_0} \right] \delta(p-\hat{p}) \end{aligned} \tag{2.54}$$

and

$$
\begin{aligned}
P_2(p,t,t_0) = &-\frac{i}{\eta}\left(\frac{e^{i2\pi pv(t-t_0)} - e^{-i2\pi pv(t-t_0)}}{2i}\right)\left[E_y^+ e^{-i2\pi\hat{p}vt_0} + E_y^- e^{i2\pi\hat{p}vt_0}\right]\delta(p-\hat{p}) \\
&+ \left(\frac{e^{i2\pi pv(t-t_0)} + e^{-i2\pi pv(t-t_0)}}{2}\right)\left[H_z^+ e^{-i2\pi\hat{p}vt_0} + H_z^- e^{i2\pi\hat{p}vt_0}\right]\delta(p-\hat{p}).
\end{aligned}
\tag{2.55}
$$

Applying an inverse Fourier transform to equations (2.54) and (2.55) yields

$$
\begin{aligned}
P_1(x,t,t_0) = &\left(\frac{e^{i2\pi\hat{p}v(t-t_0)} + e^{-i\hat{p}v(t-t_0)}}{2}\right)\left[E_y^+ e^{i2\pi\hat{p}(x-vt_0)} + E_y^- e^{i2\pi\hat{p}(x+vt_0)}\right] \\
&- i\eta\left(\frac{e^{i2\pi\hat{p}v(t-t_0)} - e^{-i2\pi\hat{p}v(t-t_0)}}{2i}\right)\left[H_z^+ e^{i2\pi\hat{p}(x-vt_0)} + H_z^- e^{i2\pi\hat{p}(x+vt_0)}\right]
\end{aligned}
\tag{2.56}
$$

and

$$
\begin{aligned}
P_2(x,t,t_0) = &-\frac{i}{\eta}\left(\frac{e^{i2\pi\hat{p}v(t-t_0)} - e^{-i2\pi\hat{p}v(t-t_0)}}{2i}\right)\left[E_y^+ e^{i2\pi\hat{p}(x-vt_0)} + E_y^- e^{i2\pi\hat{p}(x+vt_0)}\right] \\
&+ \left(\frac{e^{i2\pi\hat{p}v(t-t_0)} + e^{-i2\pi\hat{p}v(t-t_0)}}{2}\right)\left[H_z^+ e^{i2\pi\hat{p}(x-vt_0)} + H_z^- e^{i2\pi\hat{p}(x+vt_0)}\right].
\end{aligned}
\tag{2.57}
$$

Finally, using the identities from (2.51) to simplify the expressions above produces

$$
\begin{bmatrix} P_1(x,t,t_0) \\ P_2(x,t,t_0) \end{bmatrix} = \begin{bmatrix} E_y^+ \\ H_z^+ \end{bmatrix} e^{i2\pi\hat{p}(x-vt)} + \begin{bmatrix} E_y^- \\ H_z^- \end{bmatrix} e^{i2\pi\hat{p}(x+vt)} = \begin{bmatrix} E_y(x,t) \\ H_z(x,t) \end{bmatrix}.
\tag{2.58}
$$

This is indeed the plane wave at time t. As claimed, if the solution at time t_0 is available, then the propagating matrix can be used to obtain the solution at time $t > t_0$.

2.1.4 Application to One-Dimensional Transmission Lines

In this section, it is shown that the method just presented for solving the one-dimensional equation can also be applied to one-dimensional transmission lines. This is because the partial differential equations that govern the behavior of one-dimensional waves and those of a transmission line are nearly the same, albeit they use different parameters.

A one-dimensional two-port transmission line is described by the equations

$$\frac{\partial}{\partial t}V(x,t) = -\frac{1}{C(x)}\left[\frac{\partial}{\partial t}I(x,t) + G(x)V(x,t)\right] \tag{2.59}$$

and

$$\frac{\partial}{\partial t}I(x,t) = -\frac{1}{L(x)}\left[\frac{\partial}{\partial t}V(x,t) + R(x)I(x,t)\right] \tag{2.60}$$

where

- $V(x,t)$ is the voltage (in Volts),
- $I(x,t)$ is the current (in Amperes),
- $C(x)$ is the capacitance per unit length (in Farads/meter),
- $L(x)$ is the inductance per unit length (in Henrys/meter),
- $R(x)$ is the resistance per unit length (in Ohms/meter), and
- $G(x)$ is the conductance per unit length (in Siemens/meter).

In order to make the transmission line interact with other elements, whether lumped or distributed, the equation is modified slightly to include external current and voltage sources. The modified equations are

$$\frac{\partial}{\partial t}V(x,t) = -\frac{1}{C(x)}\left[\frac{\partial}{\partial x}I(x,t) + G(x)V(x,t)\right] - \frac{1}{C(x)}J_0(x,t) \tag{2.61}$$

and

$$\frac{\partial}{\partial t} I(x,t) = -\frac{1}{L(x)}\left[\frac{\partial}{\partial x} V(x,t) + R(x) I(x,t)\right] - \frac{1}{L(x)} E_0(x,t), \qquad (2.62)$$

where $J_0(x,t)$ is the current density (in Amps/m) and $E_0(x,t)$ the electric field (in Volts/m) due to external sources. The equations can be written in matrix form as

$$\frac{\partial}{\partial t}\mathbf{g}(x,t) = \begin{bmatrix} -\frac{G(x)}{C(x)} & -\frac{1}{C(x)}\frac{\partial}{\partial x} \\ -\frac{1}{L(x)}\frac{\partial}{\partial x} & -\frac{R(x)}{L(x)} \end{bmatrix} \mathbf{g}(x,t) + \mathbf{r}(x,t), \qquad (2.63)$$

where the field vectors $\mathbf{g}(x,t)$ and $\mathbf{r}(x,t)$, are given by

$$\mathbf{g}(x,t) = \begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} \qquad (2.64)$$

and

$$\mathbf{r}(x,t) = \begin{bmatrix} -\frac{1}{C(x)} & 0 \\ 0 & -\frac{1}{L(x)} \end{bmatrix} \begin{bmatrix} J_0(x,t) \\ E_0(x,t) \end{bmatrix}. \qquad (2.65)$$

Taking the Fourier transform from space (x-domain) to inverse space (p-domain) on both sides of equation (2.63) yields

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\begin{bmatrix}V(x,t)\\I(x,t)\end{bmatrix}e^{-i2\pi px}dx=\int_{-\infty}^{\infty}\begin{bmatrix}-\dfrac{G(x)}{C(x)} & -\dfrac{1}{C(x)}\dfrac{\partial}{\partial x}\\-\dfrac{1}{L(x)}\dfrac{\partial}{\partial x} & -\dfrac{R(x)}{L(x)}\end{bmatrix}\begin{bmatrix}V(x,t)\\I(x,t)\end{bmatrix}e^{-i2\pi px}dx$$
$$+\int_{-\infty}^{\infty}\begin{bmatrix}-\dfrac{1}{C(x)} & 0\\0 & -\dfrac{1}{L(x)}\end{bmatrix}\begin{bmatrix}J_0(x,t)\\E_0(x,t)\end{bmatrix}e^{-i2\pi px}dx. \tag{2.66}$$

Moving the time differential operator $\partial/\partial t$ outside of the integral and making the substitutions $F_1(x)=-G(x)/C(x)$, $F_2(x)=-1/C(x)$, $F_3(x)=-1/L(x)$, and $F_4(x)=-R(x)/L(x)$, yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\begin{bmatrix}V(x,t)\\I(x,t)\end{bmatrix}e^{-i2\pi px}dx=\int_{-\infty}^{\infty}\begin{bmatrix}F_1(x) & F_2(x)\dfrac{\partial}{\partial x}\\F_3(x)\dfrac{\partial}{\partial x} & F_4(x)\end{bmatrix}\begin{bmatrix}V(x,t)\\I(x,t)\end{bmatrix}e^{-i2\pi px}dx$$
$$+\int_{-\infty}^{\infty}\begin{bmatrix}F_2(x) & 0\\0 & F_3(x)\end{bmatrix}\begin{bmatrix}J_0(x,t)\\E_0(x,t)\end{bmatrix}e^{-i2\pi px}dx. \tag{2.67}$$

Using properties of the Fourier transform, equation (2.67) becomes

$$\frac{\partial}{\partial t}V(p,t)=\int_{-\infty}^{\infty}F_1(p-s)V(s,t)ds+\int_{-\infty}^{\infty}i2\pi sF_2(p-s)I(s,t)ds$$
$$+\int_{-\infty}^{\infty}F_2(p-s)J_0(s,t)ds \tag{2.68}$$

and

$$\frac{\partial}{\partial t}I(p,t)=\int_{-\infty}^{\infty}i2\pi sF_3(p-s)V(s,t)ds+\int_{-\infty}^{\infty}F_4(p-s)I(s,t)ds$$
$$+\int_{-\infty}^{\infty}F_3(p-s)E_0(s,t)ds. \tag{2.69}$$

Equations (2.68) and (2.69) are integro-differential equations whose general solutions are far from obvious. For the special case of a *homogeneous* transmission line, where R, C, G, and L are constants, $F_1(p)=(-G/C)\delta(p)$, $F_2(p)=(-1/C)\delta(p)$, $F_3(p)=(-1/L)\delta(p)$, and $F_4(x)=(-R/L)\delta(p)$. In this case equations (2.68) and (2.69) become

$$\frac{\partial}{\partial t}V(p,t)=\frac{-G}{C}V(p,t)-\frac{i2\pi p}{C}I(p,t)-\frac{1}{C}J_0(p,t) \tag{2.70}$$

and

$$\frac{\partial}{\partial t}I(p,t)=-\frac{i2\pi p}{L}V(p,t)-\frac{R}{L}I(p,t)-\frac{1}{L}E_0(p,t). \tag{2.71}$$

These equations can be written in matrix form as

$$\frac{\partial \mathbf{G}(p,t)}{\partial t}=\mathbf{P}(p)\mathbf{G}(p,t)+\mathbf{R}(p,t), \tag{2.72}$$

where

$$\mathbf{P}(p)=\begin{bmatrix} -\dfrac{G}{C} & -\dfrac{i2\pi p}{C} \\ -\dfrac{i2\pi p}{L} & -\dfrac{R}{L} \end{bmatrix}, \tag{2.73}$$

$$\mathbf{G}(p,t)=\begin{bmatrix} V(p,t) \\ I(p,t) \end{bmatrix}, \tag{2.74}$$

and

$$\mathbf{R}(p,t)=\begin{bmatrix} -\dfrac{1}{C}J_0(p,t) \\ -\dfrac{1}{L}E_0(p,t) \end{bmatrix}. \tag{2.75}$$

The solution to equation (2.72) at time t is the one-dimensional version of equation (2.13) given by

$$\mathbf{G}(p,t)=\mathbf{H}(p,t-t_0)\mathbf{G}(p,t_0)+\int_{t_0}^{t}\mathbf{H}(p,t-\tau)\mathbf{R}(p,\tau)\,d\tau\,, \tag{2.76}$$

where $\mathbf{G}(p,t_0)$ is the solution at time t_0.

For the special case of a lossless transmission line, where $R=0$ and $G=0$, equation (2.73) is given by

$$\mathbf{P}(p)=\begin{bmatrix} 0 & -\dfrac{i2\pi p}{C} \\ -\dfrac{i2\pi p}{L} & 0 \end{bmatrix}. \tag{2.77}$$

The kernel matrix $\mathbf{H}(p,t)$ in this case is obtained from the one-dimensional kernel matrix of equation (2.34) by making the appropriate substitutions, yielding

$$\mathbf{H}(p,t)=e^{\mathbf{P}(p)t}=\begin{bmatrix} \cos(2\pi pvt) & -iZ_0\sin(2\pi pvt) \\ -\dfrac{i}{Z_0}\sin(2\pi pvt) & \cos(2\pi pvt) \end{bmatrix}, \tag{2.78}$$

where $v=1/\sqrt{LC}$, and $Z_0=\sqrt{L/C}$.

Applying the inverse Fourier transform to equation (2.76) produces

$$\mathbf{g}(x,t)=\int_{-\infty}^{\infty}\mathbf{h}(x-\hat{x},t-t_0)\mathbf{g}(\hat{x},t_0)\,d\hat{x}+\int_{t_0}^{t}\int_{-\infty}^{\infty}\mathbf{h}(x-\hat{x},t-\tau)\mathbf{r}(\hat{x},\tau)\,d\hat{x}\,d\tau\,, \tag{2.79}$$

where the forcing function $\mathbf{r}(x,t)$, initial conditions $\mathbf{g}(x,t_0)$, and kernel matrix $\mathbf{h}(x,t)$ are given by

$$\mathbf{r}(x,t)=\begin{bmatrix} -\frac{1}{C}J_0(x,t) \\ -\frac{1}{L}E_0(x,t) \end{bmatrix}, \tag{2.80}$$

$$\mathbf{g}(x,t_0)=\begin{bmatrix} V(x,t_0) \\ I(x,t_0) \end{bmatrix}, \tag{2.81}$$

and

$$\begin{aligned}\mathbf{h}(x,t)&=\int_{-\infty}^{\infty}\begin{bmatrix} \cos(2\pi pvt) & -iZ_0\sin(2\pi pvt) \\ -\frac{i}{Z_0}\sin(2\pi pvt) & \cos(2\pi pvt) \end{bmatrix}e^{ipx}dp \\ &=\begin{bmatrix} \frac{1}{2}\left[\delta(x+vt)+\delta(x-vt)\right] & -\frac{Z_0}{2}\left[\delta(x+vt)-\delta(x-vt)\right] \\ -\frac{1}{2Z_0}\left[\delta(x+vt)-\delta(x-vt)\right] & \frac{1}{2}\left[\delta(x+vt)+\delta(x-vt)\right] \end{bmatrix}.\end{aligned} \tag{2.82}$$

Substituting the kernel matrix $\mathbf{h}(x,t)$ into equation (2.79) produces

$$\begin{aligned}V(x,t)&=\frac{1}{2}\left[V\left(x+v(t-t_0),t_0\right)+V\left(x-v(t-t_0),t_0\right)\right] \\ &-\frac{Z_0}{2}\left[I\left(x+v(t-t_0),t_0\right)-I\left(x-v(t-t_0),t_0\right)\right] \\ &-\int_0^t\frac{1}{2C}\left[J_0\left(x+v(t-\tau),\tau\right)+J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\ &+\int_0^t\frac{Z_0}{2L}\left[E_0\left(x+v(t-\tau),\tau\right)-E_0\left(x-v(t-\tau),\tau\right)\right]d\tau\end{aligned} \tag{2.83}$$

and

$$
\begin{aligned}
I(x,t) = &-\frac{1}{2Z_0}\left[V\left(x+v(t-t_0),t_0\right)-V\left(x-v(t-t_0),t_0\right)\right] \\
&+ \frac{1}{2}\left[I\left(x+v(t-t_0),t_0\right)+I\left(x-v(t-t_0),t_0\right)\right] \\
&+ \int_0^t \frac{1}{2Z_0C}\left[J_0\left(x+v(t-\tau),\tau\right)-J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\
&- \int_0^t \frac{1}{2L}\left[E_0\left(x+v(t-\tau),\tau\right)+E_0\left(x-v(t-\tau),\tau\right)\right]d\tau
\end{aligned}
\tag{2.84}
$$

Expressions (2.83) and (2.84) are exact analytic solutions for the one-dimensional lossless transmission line. Given times t_0 and t, where $t > t_0$, if the current and voltage are known at t_0 and the current density and electric field sources are known from t_0 to t, these equations can be used to compute the electric and magnetic fields at time t.

2.2 Two-Dimensional Equation

In this section the solution for the two-dimensional form of Maxwell's equations is derived. A transverse-magnetic (TM) traveling wave with components $E_z(x,y,t)$, $H_x(x,y,t)$, and $H_y(x,y,t)$ is used for the derivation, but the same analysis can be applied without difficulty to a transverse-electric (TE) wave. The TM wave propagates according to

$$\frac{\partial}{\partial t}\mathbf{g}(x,y,t)=\hat{\mathbf{P}}(x,y)\mathbf{g}(x,y,t)+\mathbf{r}(x,y,t), \tag{2.85}$$

where

$$\mathbf{g}(x,y,t)=\begin{bmatrix} E_z(x,y,t) \\ H_x(x,y,t) \\ H_y(x,y,t) \end{bmatrix}, \tag{2.86}$$

$$\hat{\mathbf{P}}(x,y)=\begin{bmatrix} -\frac{\sigma_e(x,y)}{\varepsilon(x,y)} & -\frac{1}{\varepsilon(x,y)}\frac{\partial}{\partial y} & \frac{1}{\varepsilon(x,y)}\frac{\partial}{\partial x} \\ -\frac{1}{\mu(x,y)}\frac{\partial}{\partial y} & -\frac{\sigma_m(x,y)}{\mu(x,y)} & 0 \\ \frac{1}{\mu(x,y)}\frac{\partial}{\partial x} & 0 & -\frac{\sigma_m(x,y)}{\mu(x,y)} \end{bmatrix}, \tag{2.87}$$

and

$$\mathbf{r}(x,y,t)=\begin{bmatrix} -\frac{1}{\varepsilon(x,y)} & 0 & 0 \\ 0 & -\frac{1}{\mu(x,y)} & 0 \\ 0 & 0 & -\frac{1}{\mu(x,y)} \end{bmatrix}\begin{bmatrix} J_z(x,y,t) \\ M_x(x,y,t) \\ M_y(x,y,t) \end{bmatrix}. \tag{2.88}$$

To transform from real- to inverse-space, the Fourier transform from the (x,y) domain to the (p,q) domain is applied on both sides of equation (2.88). Using the substitutions $F_1(x,y) = -\sigma_e(x,y)/\varepsilon(x,y)$, $F_2(x,y) = -\sigma_m(x,y)/\mu(x,y)$, $F_3(x,y) = 1/\varepsilon(x,y)$, and $F_4(x,y) = 1/\mu(x,y)$, this operation yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}(x,y,t)e^{-i2\pi(px+qy)}dxdy =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} F_1(x,y) & -F_3(x,y)\frac{\partial}{\partial y} & F_3(x,y)\frac{\partial}{\partial x} \\ -F_4(x,y)\frac{\partial}{\partial y} & F_2(x,y) & 0 \\ F_4(x,y)\frac{\partial}{\partial x} & 0 & F_2(x,y) \end{bmatrix}\mathbf{g}(x,y,t)e^{-i2\pi(px+qy)}dxdy. \quad (2.89)$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} -F_3(x,y) & 0 & 0 \\ 0 & -F_4(x,y) & 0 \\ 0 & 0 & -F_4(x,y) \end{bmatrix}\begin{bmatrix} J_z(x,y,t) \\ M_x(x,y,t) \\ M_y(x,y,t) \end{bmatrix}e^{-i2\pi(px+qy)}dxdy$$

Using the following properties of the Fourier transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}m(x,y)n(x,y)e^{-i2\pi(px+qy)}dxdy =$$
$$M(p,q)*N(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}M(p-\hat{p},q-\hat{q})N(\hat{p},\hat{q})d\hat{p}\,d\hat{q}, \quad (2.90)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}m(x,y)\right\}e^{-i2\pi(px+qy)}dxdy = i2\pi pM(p,q), \quad (2.91)$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial y}m(x,y)\right\}e^{-i2\pi(px+qy)}dxdy = i2\pi qM(p,q), \quad (2.92)$$

and applying them to equation (2.89) produces the following three equations

$$\begin{aligned}\frac{\partial}{\partial t}E_z(p,q,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(p-\hat{p},q-\hat{q})E_z(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_3(p-\hat{p},q-\hat{q})H_x(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_3(p-\hat{p},q-\hat{q})H_y(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_3(p-\hat{p},q-\hat{q})J_z(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q},\end{aligned} \tag{2.93}$$

$$\begin{aligned}\frac{\partial}{\partial t}H_x(p,q,t) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_4(p-\hat{p},q-\hat{q})E_z(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(p-\hat{p},q-\hat{q})H_x(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_4(p-\hat{p},q-\hat{q})M_x(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q},\end{aligned} \tag{2.94}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}H_y(p,q,t) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_4(p-\hat{p},q-\hat{q})E_z(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(p-\hat{p},q-\hat{q})H_y(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q} \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_4(p-\hat{p},q-\hat{q})M_y(\hat{p},\hat{q},t)\,d\hat{p}\,d\hat{q}.\end{aligned} \tag{2.95}$$

Equations (2.93) to (2.95) are integro-differential equations for which an analytic solution may be difficult or even impossible to obtain. If the medium is homogeneous, however, a solution can indeed be obtained.

2.2.1 Solution to the Two-Dimensional Equation in a Homogeneous Medium

In the special case of a *homogeneous* medium where ε, μ, σ, and ρ are constants, $F_1(p,q)=(-\sigma_e/\varepsilon)\delta(p,q)$, $F_2(p,q)=(-\sigma_m/\mu)\delta(p,q)$, $F_3(p,q)=(1/\varepsilon)\delta(p,q)$, and $F_4(p,q)=(1/\mu)\delta(p,q)$. In this case equations (2.93) to (2.95) become

$$\frac{\partial \mathbf{G}(p,q,t)}{\partial t}=\mathbf{P}(p,q)\mathbf{G}(p,q,t)+\mathbf{R}(p,q,t), \tag{2.96}$$

where

$$\mathbf{P}(p,q)=\begin{bmatrix} -\frac{\sigma_e}{\varepsilon} & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} \\ -\frac{i2\pi q}{\mu} & -\frac{\sigma_m}{\mu} & 0 \\ \frac{i2\pi p}{\mu} & 0 & -\frac{\sigma_m}{\mu} \end{bmatrix}, \tag{2.97}$$

$$\mathbf{G}(p,q,t)=\begin{bmatrix} E_z(p,q,t) \\ H_x(p,q,t) \\ H_y(p,q,t) \end{bmatrix}, \tag{2.98}$$

and

$$\mathbf{R}(p,q,t)=\begin{bmatrix} -\frac{1}{\varepsilon}J_z(p,q,t) \\ -\frac{1}{\mu}M_x(p,q,t) \\ -\frac{1}{\mu}M_y(p,q,t) \end{bmatrix}. \tag{2.99}$$

The solution to equation (2.96) at time t is the two-dimensional version of equation (2.13) given by

$$\mathbf{G}(p,q,t)=\mathbf{H}(p,q,t-t_0)\mathbf{G}(p,q,t_0)+\int_{t_0}^{t}\mathbf{H}(p,q,t-\tau)\mathbf{R}(p,q,\tau)\,d\tau, \qquad (2.100)$$

where $\mathbf{G}(p,q,t_0)$ is the solution at time t_0.

In the special case of a two-dimensional homogeneous medium $\sigma_e=0$, $\sigma_m=0$, and ε and μ are constants. In this case the matrix of equation (2.97) is

$$\mathbf{P}(p,q)=\begin{bmatrix} 0 & -\dfrac{i2\pi q}{\varepsilon} & \dfrac{i2\pi p}{\varepsilon} \\ -\dfrac{i2\pi q}{\mu} & 0 & 0 \\ \dfrac{i2\pi p}{\mu} & 0 & 0 \end{bmatrix}. \qquad (2.101)$$

The corresponding kernel matrix $\mathbf{H}(p,q,t)$ for this case (see appendix B) is

$$\mathbf{H}(p,q,t)=\begin{bmatrix} 1-k^2F & -i\eta qG & i\eta pG \\ \dfrac{-iq}{\eta}G & 1-q^2F & pqF \\ \dfrac{ip}{\eta}G & pqF & 1-p^2F \end{bmatrix}, \qquad (2.102)$$

where $F=\left(1-\cos(2\pi kvt)\right)/k^2$, $G=\sin(2\pi kvt)/k$, $k=\sqrt{p^2+q^2}$, $v=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.

Substituting $\mathbf{H}(p,q,t)$ from (2.102) into equation (2.100) and applying an inverse Fourier transform, yields the following three equations:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_z(p,q,t)\,e^{i2\pi(px+qy)}dp\,dq = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_z(p,q,t_0)\,e^{i2\pi(px+qy)}dp\,dq \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon(t-t_0)\right)}{k^2}\right)\left[(i2\pi p)^2+(i2\pi q)^2\right]E_z(p,q,t_0)\,e^{i2\pi(px+qy)}dp\,dq \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon(t-t_0)\right)}{k}\right)i2\pi q\eta H_x(p,q,t_0)\,e^{i2\pi(px+qy)}dp\,dq \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon(t-t_0)\right)}{k}\right)i2\pi p\eta H_y(p,q,t_0)\,e^{i2\pi(px+qy)}dp\,dq \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}J_z(p,q,\tau)\,e^{i2\pi(px+qy)}dp\,dq\,dt \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon(t-\tau)\right)}{k^2}\right)\left[(i2\pi p)^2+(i2\pi q)^2\right]\frac{1}{\varepsilon}J_z(p,q,\tau)\,e^{i2\pi(px+qy)}dp\,dq\,dt \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon(t-\tau)\right)}{k}\right)i2\pi q\frac{\eta}{\mu}M_x(p,q,\tau)\,e^{i2\pi(px+qy)}dp\,dq\,dt \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon(t-\tau)\right)}{k}\right)i2\pi p\frac{\eta}{\mu}M_y(p,q,\tau)\,e^{i2\pi(px+qy)}dp\,dq\,dt,
\end{aligned}
\tag{2.103}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_x(p,q,t)\, e^{i(px+qy)}\,dp\,dq = \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon\left(t-t_0\right)\right)}{k}\right) i2\pi q\,\frac{1}{\eta}E_z\left(p,q,t_0\right)e^{i2\pi(px+qy)}\,dp\,dq \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_x\left(p,q,t_0\right)e^{i(px+qy)}\,dp\,dq \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon\left(t-t_0\right)\right)}{k^2}\right)\left(i2\pi q\right)^2 H_x\left(p,q,t_0\right)e^{i2\pi(px+qy)}\,dp\,dq \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon\left(t-t_0\right)\right)}{k^2}\right)\left(i2\pi p\right)\left(i2\pi q\right)H_y\left(p,q,t_0\right)e^{i2\pi(px+qy)}\,dp\,dq \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi k\upsilon\left(t-\tau\right)\right)}{k}\right) i2\pi q\,\frac{1}{\eta\varepsilon}J_z\left(p,q,\tau\right)e^{i2\pi(px+qy)}\,dp\,dq\,dt \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu}M_x\left(p,q,\tau\right)e^{i2\pi(px+qy)}\,dp\,dq\,dt \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon\left(t-\tau\right)\right)}{k^2}\right)\left(i2\pi q^2\right)\frac{1}{\mu}M_x\left(p,q,\tau\right)e^{i2\pi(px+qy)}\,dp\,dq\,dt \qquad (2.104) \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi k\upsilon\left(t-\tau\right)\right)}{k^2}\right)\left(i2\pi p\right)\left(i2\pi q\right)\frac{1}{\mu}M_y\left(p,q,\tau\right)e^{i2\pi(px+qy)}\,dp\,dq\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_y(p,q,t)e^{i2\pi(px+qy)}dp\,dq = \\
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi k\upsilon(t-t_0))}{k}\right) i2\pi p\frac{1}{\eta}E_z(p,q,t_0)e^{i2\pi(px+qy)}dp\,dq \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi k\upsilon(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi q)H_x(p,q,t_0)e^{i2\pi(px+qy)}dp\,dq \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_y(p,q,t_0)e^{i2\pi(px+qy)}dp\,dq \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi k\upsilon(t-t_0))}{k^2}\right)(i2\pi p)^2 H_y(p,q,t_0)e^{i2\pi(px+qy)}dp\,dq \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi k\upsilon(t-\tau))}{k}\right) i2\pi p\frac{1}{\eta\varepsilon}J_z(p,q,\tau)e^{i2\pi(px+qy)}dp\,dq\,dt \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi k\upsilon(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi q)\frac{1}{\mu}M_x(p,q,\tau)e^{i2\pi(px+qy)}dp\,dq\,dt \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu}M_y(p,q,\tau)e^{i2\pi(px+qy)}dp\,dq\,dt \qquad (2.105) \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi k\upsilon(t-t_0))}{k^2}\right)(i2\pi p)^2\frac{1}{\mu}M_y(p,q,\tau)e^{i2\pi(px+qy)}dp\,dq\,dt.
\end{aligned}
$$

Using the differentiation properties of the Fourier transform, equations (2.103) to (2.105) become

$$
\begin{aligned}
&E_z(x,y,t) = E_z(x,y,t_0) + F_1(x,y,t-t_0) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z(x,y,t_0) \\
&-\eta F_2(x,y,t-t_0) * \frac{\partial}{\partial y} H_x(x,y,t_0) + \eta F_2(x,y,t-t_0) * \frac{\partial}{\partial x} H_y(x,y,t_0) \\
&-\int_{t_0}^{t} \left[F_1(x,y,t-\tau) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{1}{\varepsilon} J_z(x,y,\tau) \right] d\tau - \int_{t_0}^{t} \frac{1}{\varepsilon} J_z(x,y,\tau)\, d\tau \\
&+\int_{t_0}^{t} \left[\eta F_2(x,y,t-\tau) * \frac{\partial}{\partial y} \frac{1}{\mu} M_x(x,y,\tau) \right] d\tau \\
&-\int_{t_0}^{t} \left[\eta F_2(x,y,t-\tau) * \frac{\partial}{\partial x} \frac{1}{\mu} M_y(x,y,\tau) \right] d\tau,
\end{aligned}
\tag{2.106}
$$

$$
\begin{aligned}
&H_x(x,y,t) = -\frac{1}{\eta} F_2(x,y,t-t_0) * \frac{\partial}{\partial y} E_z(x,y,t_0) \\
&+H_x(x,y,t_0) + F_1(x,y,t-t_0) * \frac{\partial^2}{\partial y^2} H_x(x,y,t_0) \\
&-F_1(x,y,t-t_0) * \frac{\partial^2}{\partial x \partial y} H_y(x,y,t_0) \\
&+\int_{t_0}^{t} \left[\frac{1}{\eta} F_2(x,y,t-\tau) * \frac{\partial}{\partial y} \frac{1}{\varepsilon} J_z(x,y,\tau) \right] d\tau \\
&-\int_{t_0}^{t} \frac{1}{\mu} M_x(x,y,\tau)\, d\tau - \int_{t_0}^{t} \left[F_1(x,y,t-\tau) * \frac{\partial^2}{\partial y^2} \frac{1}{\mu} M_x(x,y,\tau) \right] d\tau \\
&+\int_{t_0}^{t} \left[F_1(x,y,t-\tau) * \frac{\partial^2}{\partial x \partial y} \frac{1}{\mu} M_y(x,y,\tau) \right] d\tau,
\end{aligned}
\tag{2.107}
$$

and

$$
\begin{aligned}
H_y(x,y,t) &= \frac{1}{\eta}F_2(x,y,t-t_0)*\frac{\partial}{\partial x}E_z(x,y,t_0) \\
&-F_1(x,y,t-t_0)*\frac{\partial^2}{\partial x\partial y}H_x(x,y,t_0) \\
&+H_y(x,y,t_0)+F_1(x,y,t-t_0)*\frac{\partial^2}{\partial x^2}H_y(x,y,t_0) \\
&-\int_{t_0}^{t}\left[\frac{1}{\eta}F_2(x,y,t-\tau)*\frac{\partial}{\partial x}\frac{1}{\varepsilon}J_z(x,y,\tau)\right]d\tau \\
&+\int_{t_0}^{t}\left[F_1(x,y,t-\tau)*\frac{\partial^2}{\partial x\partial y}\frac{1}{\mu}M_x(x,y,\tau)\right]d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_y(x,y,\tau)d\tau-\int_{t_0}^{t}\left[F_1(x,y,t-\tau)*\frac{\partial^2}{\partial x^2}\frac{1}{\mu}M_y(x,y,\tau)\right]d\tau,
\end{aligned}
\tag{2.108}
$$

where $*$ denotes convolution and

$$F_1(x,y,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kvt)}{k^2}\right)e^{i2\pi(px+qy)}dp\,dq \tag{2.109}$$

and

$$F_2(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kvt)}{k}\right)e^{i2\pi(px+qy)}dp\,dq\,. \tag{2.110}$$

In order to obtain closed-form expressions for the integrals in equations (2.109) and (2.110), the coordinate system has to be changed. Switching to polar coordinates, equations (2.109) and (2.110) become

$$F_1(r,t)=\frac{1}{(2\pi)^2}\int_{k=0}^{\infty}\int_{\phi=0}^{2\pi}\left(\frac{1-\cos(2\pi kvt)}{k^2}\right)e^{ikr\cos(\theta-\phi)}k\,dk\,d\phi \tag{2.111}$$

and

$$F_2(r,t)=\frac{1}{2\pi}\int_{k=0}^{\infty}\int_{\phi=0}^{2\pi}\left(\frac{\sin(2\pi kvt)}{k}\right)e^{ikr\cos(\theta-\phi)}k\,dk\,d\phi\,, \tag{2.112}$$

where $x = r\cos\theta$, $y = r\sin\theta$, $p = k\cos\phi$, and $q = k\sin\phi$. Using the identity

$$J_0(x) = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} e^{ix\cos\varphi}d\varphi, \tag{2.113}$$

where J_0 denotes the Bessel function of order zero, expressions (2.111) and (2.112) can be written as

$$F_1(r,t) = \frac{1}{2\pi}\int_{k=0}^{\infty}\left(\frac{1-\cos(2\pi kvt)}{k}\right)J_0(2\pi kr)dk \tag{2.114}$$

and

$$F_2(r,t) = \int_{k=0}^{\infty}\sin(2\pi kvt)J_0(2\pi kr)dk. \tag{2.115}$$

Closed-form expressions are obtained using a table of integrals [4, 6.671-7 and 6.696] and are given below

$$F_1(x,y,t) = \begin{cases} \dfrac{1}{2\pi}\ln\left(\dfrac{vt+\sqrt{(vt)^2-x^2-y^2}}{\sqrt{x^2+y^2}}\right) & , x^2+y^2<(vt)^2 \\ 0 & , else \end{cases} \tag{2.116}$$

and

$$F_2(x,y,t) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{\sqrt{(vt)^2-x^2-y^2}} & , x^2+y^2<(vt)^2 \\ 0 & , else \end{cases} \tag{2.117}$$

Returning to equations (2.106) to (2.108) and writing the convolutions explicitly yields

$$
\begin{aligned}
&E_z(x,y,t) = E_z(x,y,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{y}^2}\right)E_z(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\,\eta\frac{\partial}{\partial\hat{y}}H_x(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\,\eta\frac{\partial}{\partial\hat{x}}H_y(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&-\int_{t_0}^{t}\frac{1}{\varepsilon}J_z(x,y,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{y}^2}\right)J_z(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{y}}M_x(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{x}}M_y(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau,
\end{aligned}
\tag{2.118}
$$

$$
\begin{aligned}
&H_x(x,y,t) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{y}}E_z(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&+H_x(x,y,t_0)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\frac{\partial^2}{\partial\hat{y}^2}H_x(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}H_y(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{y}}J_z(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_x(x,y,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{y}^2}M_x(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}M_y(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau,
\end{aligned}
\tag{2.119}
$$

and

$$
\begin{aligned}
H_y(x,y,t) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{x}}E_z(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}H_x(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&+H_y(x,y,t_0)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\frac{\partial^2}{\partial\hat{x}^2}H_y(\hat{x},\hat{y},t_0)\,d\hat{x}\,d\hat{y} \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{x}}J_z(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}M_x(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_y(x,y,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}^2}M_y(\hat{x},\hat{y},\tau)\,d\hat{x}\,d\hat{y}\,d\tau.
\end{aligned}
\tag{2.120}
$$

Expressions (2.118) to (2.120) are exact analytic solutions to the two-dimensional form of Maxwell's equations in a lossless homogeneous medium. Given times t_0 and t, where $t > t_0$, if the electric and magnetic fields are known at t_0 and the current and magnetic sources are known from t_0 to t, these equations can be used to compute the electric and magnetic fields at time t.

2.2.2 Time Evolution of Two-Dimensional Plane Waves

In this section the results obtained in the previous section area applied to a two-dimensional plane wave. In a homogeneous medium with $(\mathbf{J}_{source} = \mathbf{M}_{source} = 0)$ and $\sigma_e = \sigma_m = 0$, the two-dimensional Maxwell's curl equations are

$$\frac{\partial}{\partial t}E_z(x,y,t) = -\frac{1}{\varepsilon}\frac{\partial}{\partial y}H_x(x,y,t) + \frac{1}{\varepsilon}\frac{\partial}{\partial x}H_y(x,y,t), \tag{2.121}$$

$$\frac{\partial}{\partial t}H_x(x,y,t) = -\frac{1}{\mu}\frac{\partial}{\partial y}E_z(x,y,t), \tag{2.122}$$

and

$$\frac{\partial}{\partial t}H_y(x,y,t) = \frac{1}{\mu}\frac{\partial}{\partial x}E_z(x,y,t). \tag{2.123}$$

A two-dimensional plane wave with wavenumber $\hat{k}$ propagating in the xy-plane at an angle θ having components $E_z(x,y,t)$, $H_x(x,y,t)$, and $H_y(x,y,t)$ is given by

$$\begin{bmatrix} E_z(x,y,t) \\ H_x(x,y,t) \\ H_y(x,y,t) \end{bmatrix} = \begin{bmatrix} E_z^+ \\ H_x^+ \\ H_y^+ \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta + y\sin\theta - vt)} + \begin{bmatrix} E_z^- \\ H_x^- \\ H_y^- \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta + y\sin\theta + vt)}. \tag{2.124}$$

Substituting (2.124) into equations (2.121) to (2.123) yields

$$\begin{aligned} &-i2\pi\hat{k}vE_z^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}vE_z^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)} = \\ &\quad -\frac{1}{\varepsilon}\left[i2\pi\hat{k}\sin\theta H_x^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}\sin\theta H_x^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)}\right] \\ &\quad +\frac{1}{\varepsilon}\left[i2\pi\hat{k}\cos\theta H_y^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}\cos\theta H_y^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)}\right], \end{aligned} \tag{2.125}$$

$$\begin{aligned} &-i2\pi\hat{k}vH_x^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}vH_x^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)} = \\ &\quad -\frac{1}{\mu}\left[i2\pi\hat{k}\sin\theta E_z^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}\sin\theta E_z^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)}\right], \end{aligned} \tag{2.126}$$

and

$$\begin{aligned} &-i2\pi\hat{k}vH_y^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}vH_y^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)} = \\ &\quad \frac{1}{\mu}\left[i2\pi\hat{k}\cos\theta E_z^+e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + i2\pi\hat{k}\cos\theta E_z^-e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)}\right]. \end{aligned} \tag{2.127}$$

Equating the common terms in equations (2.125) to (2.127) yields

$$E_z^+ = \frac{\sin\theta}{\varepsilon v} H_x^+ - \frac{\cos\theta}{\varepsilon v} H_y^+ \qquad E_z^- = -\frac{\sin\theta}{\varepsilon v} H_x^- + \frac{\cos\theta}{\varepsilon v} H_y^-, \tag{2.128}$$

$$H_x^+ = \frac{\sin\theta}{\mu v} E_z^+ \qquad H_x^- = -\frac{\sin\theta}{\mu v} E_z^-, \tag{2.129}$$

and

$$H_y^+ = -\frac{\cos\theta}{u v} E_z^+ \qquad H_y^- = \frac{\cos\theta}{u v} E_z^-. \tag{2.130}$$

As expected, it is required that $v = 1/\sqrt{\mu\varepsilon}$. Substituting v into equations (2.128) and (2.130) produces the following identities

$$E_z^+ = \eta \sin\theta H_x^+ - \eta \cos\theta H_y^+ \qquad E_z^- = -\eta \sin\theta H_x^- + \eta \cos\theta H_y^-, \tag{2.131}$$

$$H_x^+ = \frac{\sin\theta}{\eta} E_z^+ \qquad H_x^- = -\frac{\sin\theta}{\eta} E_z^-, \tag{2.132}$$

and

$$H_y^+ = -\frac{\cos\theta}{\eta} E_z^+ \qquad H_y^- = \frac{\cos\theta}{\eta} E_z^-. \tag{2.133}$$

These identities are used in the coming sections when proving that the results obtained thus far, applied to a plane wave, correctly predict its propagation. In the next two sections, an indirect method using the kernel matrix from (2.102) and a direct method using expressions (2.118) to (2.120) are used to independently show that this is indeed the case.

2.2.2.1 Time Evolution of Two-Dimensional Plane Waves: Indirect Approach

In this section it is shown that in a source-free lossless homogeneous medium, application of the kernel matrix $\mathbf{H}(p,q,t-t_0)$ given in (2.102) to a plane wave at time t_0 results in the plane wave at time t. Applying the Fourier transform to equation (2.124) and evaluating at time t_0 produces

$$\begin{bmatrix} E_z(p,q,t_0) \\ H_x(p,q,t_0) \\ H_y(p,q,t_0) \end{bmatrix} = \left(\begin{bmatrix} E_z^+ \\ H_x^+ \\ H_y^+ \end{bmatrix} e^{-i2\pi\hat{k}vt_0} + \begin{bmatrix} E_z^- \\ H_x^- \\ H_y^- \end{bmatrix} e^{i2\pi\hat{k}vt_0} \right) \delta\left(p-\hat{k}\cos\theta, q-\hat{k}\sin\theta\right). \qquad (2.134)$$

Next, the kernel matrix $\mathbf{H}(p,q,t-t_0)$ is applied to the plane wave of equation (2.134). If our derivations thus far have been correct, this operation should yield the same plane wave at time t. Multiplying the kernel matrix $\mathbf{H}(p,t-t_0)$ with the plane wave produces

$$\begin{bmatrix} P_1(p,q,t,t_0) \\ P_2(p,q,t,t_0) \\ P_3(p,q,t,t_0) \end{bmatrix} = \begin{bmatrix} 1-k^2F & -i\eta qG & i\eta pG \\ \frac{-iq}{\eta}G & 1-q^2F & pqF \\ \frac{ip}{\eta}G & pqF & 1-p^2F \end{bmatrix} \begin{bmatrix} E_z(p,q,t_0) \\ H_x(p,q,t_0) \\ H_y(p,q,t_0) \end{bmatrix}, \qquad (2.135)$$

where $F=\left(1-\cos(2\pi kvt)\right)/k^2$, $G=\sin(2\pi kvt)/k$, $k=\sqrt{p^2+q^2}$, $v=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.

Substituting (2.134) into equation (2.135), yields

$$
\begin{aligned}
&P_1(p,q,t,t_0)= \\
&\quad \cos\left(2\pi kv(t-t_0)\right)\left[E_z^+ e^{-i2\pi\hat{k}vt_0}+E_z^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad -\frac{i\eta q\sin\left(2\pi kv(t-t_0)\right)}{k}\left[H_x^+ e^{-i2\pi\hat{k}vt_0}+H_x^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad +\frac{i\eta p\sin\left(2\pi kv(t-t_0)\right)}{k}\left[H_y^+ e^{-i2\pi\hat{k}vt_0}+H_y^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right),
\end{aligned}
\tag{2.136}
$$

$$
\begin{aligned}
&P_2(p,q,t,t_0)= \\
&\quad \frac{-iq\sin\left(2\pi kv(t-t_0)\right)}{\eta k}\left[E_z^+ e^{-i2\pi\hat{k}vt_0}+E_z^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad +\frac{p^2+q^2\cos\left(2\pi kv(t-t_0)\right)}{k^2}\left[H_x^+ e^{-i2\pi\hat{k}vt_0}+H_x^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad +\frac{pq\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}\left[H_y^+ e^{-i2\pi\hat{k}vt_0}+H_y^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right),
\end{aligned}
\tag{2.137}
$$

and

$$
\begin{aligned}
&P_3(p,q,t,t_0)= \\
&\quad \frac{ip\sin\left(2\pi kv(t-t_0)\right)}{\eta k}\left[E_z^+ e^{-i2\pi\hat{k}vt_0}+E_z^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad +\frac{pq\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}\left[H_x^+ e^{-i2\pi\hat{k}vt_0}+H_x^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right) \\
&\quad +\frac{q^2+p^2\cos\left(2\pi kv(t-t_0)\right)}{k^2}\left[H_y^+ e^{-i2\pi\hat{k}vt_0}+H_y^- e^{i2\pi\hat{k}vt_0}\right]\delta\left(p-\hat{k}\cos\theta,q-\hat{k}\sin\theta\right).
\end{aligned}
\tag{2.138}
$$

Applying an inverse Fourier transform to equations (2.136) to (2.138) and using the identities (2.131) and (2.133) yields

$$
\begin{aligned}
P_1(x,y,t,t_0) &= \left[E_z^+ e^{-i2\pi\hat{k}vt_0} + E_z^- e^{i2\pi\hat{k}vt_0}\right]\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&- i\eta\sin\theta\left[\frac{\sin\theta}{\eta}E_z^+ e^{-i2\pi\hat{k}vt_0} - \frac{\sin\theta}{\eta}E_z^- e^{i2\pi\hat{k}vt_0}\right]\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+ i\eta\cos\theta\left[-\frac{\cos\theta}{\eta}E_z^+ e^{-i2\pi\hat{k}vt_0} + \frac{\cos\theta}{\eta}E_z^- e^{i2\pi\hat{k}vt_0}\right]\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)},
\end{aligned}
\tag{2.139}
$$

$$
\begin{aligned}
&P_2(x,y,t,t_0) = \\
&-i\frac{\sin\theta}{\eta}\left[\frac{\eta}{\sin\theta}H_x^+ e^{-i2\pi\hat{k}vt_0} - \frac{\eta}{\sin\theta}H_x^- e^{i2\pi\hat{k}vt_0}\right]\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\cos^2\theta\left[H_x^+ e^{-i2\pi\hat{k}vt_0} + H_x^- e^{i2\pi\hat{k}vt_0}\right]e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\sin^2\theta\left[H_x^+ e^{-i2\pi\hat{k}vt_0} + H_x^- e^{i2\pi\hat{k}vt_0}\right]\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\sin\theta\cos\theta\left[-\frac{\cos\theta}{\sin\theta}H_x^+ e^{-i2\pi\hat{k}vt_0} - \frac{\cos\theta}{\sin\theta}H_x^- e^{i2\pi\hat{k}vt_0}\right]e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&-\sin\theta\cos\theta\left[-\frac{\cos\theta}{\sin\theta}H_x^+ e^{-i2\pi\hat{k}vt_0} - \frac{\cos\theta}{\sin\theta}H_x^- e^{i2\pi\hat{k}vt_0}\right]\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)},
\end{aligned}
\tag{2.140}
$$

and

$$
\begin{aligned}
&P_3(x,y,t,t_0) = \\
&i\frac{\cos\theta}{\eta}\left[-\frac{\eta}{\cos\theta}H_y^+ e^{-i2\pi\hat{k}vt_0} + \frac{\eta}{\cos\theta}H_y^- e^{i2\pi\hat{k}vt_0}\right]\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\sin\theta\cos\theta\left[-\frac{\sin\theta}{\cos\theta}H_y^+ e^{-i2\pi\hat{k}vt_0} - \frac{\sin\theta}{\cos\theta}H_y^- e^{i2\pi\hat{k}vt_0}\right]e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&-\sin\theta\cos\theta\left[-\frac{\sin\theta}{\cos\theta}H_y^+ e^{-i2\pi\hat{k}vt_0} - \frac{\sin\theta}{\cos\theta}H_y^- e^{i2\pi\hat{k}vt_0}\right]\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\sin^2\theta\left[H_y^+ e^{-i2\pi\hat{k}vt_0} + H_y^- e^{i2\pi\hat{k}vt_0}\right]e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)} \\
&+\cos^2\theta\left[H_y^+ e^{-i2\pi\hat{k}vt_0} + H_y^- e^{i2\pi\hat{k}vt_0}\right]\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta)}.
\end{aligned}
\tag{2.141}
$$

After some algebra equations (2.139) to (2.141) become

$$\begin{bmatrix} P_1(x,y,t,t_0) \\ P_2(x,y,t,t_0) \\ P_3(x,y,t,t_0) \end{bmatrix} = \begin{bmatrix} E_z^+ \\ H_x^+ \\ H_y^+ \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta + y\sin\theta - vt)} + \begin{bmatrix} E_z^- \\ H_x^- \\ H_y^- \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta + y\sin\theta + vt)} = \begin{bmatrix} E_z(x,y,t) \\ H_x(x,y,t) \\ H_y(x,y,t) \end{bmatrix}. \quad (2.142)$$

This result verifies the previous claim. Clearly, if the solution at time t_0 is available, the propagating matrix can be used to obtain the solution at time $t > t_0$.

2.2.2.2 Time Evolution of Two-Dimensional Plane Waves: Direct Approach

In this section it is shown that in a source-free lossless homogeneous medium, direct application of equations (2.118) to (2.120) to a two-dimensional plane wave at time t_0 produces a two-dimensional plane wave at time $t > t_0$.

Substituting equation (2.134) into equations (2.118) to (2.120), and using identities (2.131) and (2.133) produces

$$
\begin{aligned}
P_1(x,y,t,t_0) &= E_z^+ e^{i\hat{k}(x\cos\theta+y\sin\theta-vt_0)} + E_z^- e^{i\hat{k}(x\cos\theta+y\sin\theta+vt_0)} \\
&+\left[\left(i2\pi\hat{k}\cos\theta\right)^2+\left(i2\pi\hat{k}\sin\theta\right)^2\right]E_z^+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&+\left[\left(i2\pi\hat{k}\cos\theta\right)^2+\left(i2\pi\hat{k}\sin\theta\right)^2\right]E_z^-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&-\eta\left(i2\pi\hat{k}\sin\theta\right)\left(\frac{\sin\theta}{\eta}E_z^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&-\eta\left(i2\pi\hat{k}\sin\theta\right)\left(-\frac{\sin\theta}{\eta}E_z^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&+\eta\left(i2\pi\hat{k}\cos\theta\right)\left(-\frac{\cos\theta}{\eta}E_z^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&+\eta\left(i2\pi\hat{k}\cos\theta\right)\left(\frac{\cos\theta}{\eta}E_z^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y},
\end{aligned}
\tag{2.143}
$$

$$
\begin{aligned}
P_2(x,y,t,t_0) &= H_x^+ e^{i\hat{k}(x\cos\theta+y\sin\theta-vt_0)} + H_x^- e^{i\hat{k}(x\cos\theta+y\sin\theta+vt_0)} \\
&-\frac{1}{\eta}\left(i2\pi\hat{k}\sin\theta\right)\left(\frac{\eta}{\sin\theta}H_x^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&-\frac{1}{\eta}\left(i2\pi\hat{k}\sin\theta\right)\left(-\frac{\eta}{\sin\theta}H_x^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\sin\theta\right)^2 H_x^+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\sin\theta\right)^2 H_x^-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\sin\theta\right)\left(i2\pi\hat{k}\cos\theta\right)\left(-\frac{\cos\theta}{\sin\theta}H_y^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\sin\theta\right)\left(i2\pi\hat{k}\cos\theta\right)\left(-\frac{\cos\theta}{\sin\theta}H_y^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1\left(x-\hat{x},y-\hat{y},t-t_0\right)e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y},
\end{aligned}
\tag{2.144}
$$

and

$$
\begin{aligned}
&P_3(x,y,t,t_0) = H_y^+ e^{i\hat{k}(x\cos\theta + y\sin\theta - vt_0)} + H_y^- e^{i\hat{k}(x\cos\theta + y\sin\theta + vt_0)} \\
&+\frac{1}{\eta}\left(i2\pi\hat{k}\cos\theta\right)\left(-\frac{\eta}{\cos\theta}H_x^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&+\frac{1}{\eta}\left(i2\pi\hat{k}\cos\theta\right)\left(\frac{\eta}{\cos\theta}H_x^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\sin\theta\right)\left(i2\pi\hat{k}\cos\theta\right)\left(-\frac{\sin\theta}{\cos\theta}H_y^+\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\sin\theta\right)\left(i2\pi\hat{k}\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}H_y^-\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\cos\theta\right)^2 H_x^+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\cos\theta\right)^2 H_x^-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},t-t_0)\,e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}d\hat{x}\,d\hat{y}.
\end{aligned} \tag{2.145}
$$

By making a change of variables equations (2.143) to (2.145) can be rewritten as

$$
\begin{aligned}
&P_1(x,y,t,t_0) = E_z^+ e^{i\hat{k}(x\cos\theta + y\sin\theta - vt_0)} + E_z^- e^{i\hat{k}(x\cos\theta + y\sin\theta + vt_0)} \\
&-\left(2\pi\hat{k}\right)^2 E_z^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} \\
&-\left(2\pi\hat{k}\right)^2 E_z^- e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\sin^2\theta\right)E_z^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\sin^2\theta\right)E_z^- e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} \\
&-\left(i2\pi\hat{k}\cos^2\theta\right)E_z^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta-vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} \\
&+\left(i2\pi\hat{k}\cos^2\theta\right)E_z^- e^{i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta+vt_0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0)\,e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y},
\end{aligned} \tag{2.146}
$$

$$P_2(x,y,t,t_0) = H_x^+ e^{i\hat{k}(x\cos\theta + y\sin\theta - vt_0)} + H_x^- e^{i\hat{k}(x\cos\theta + y\sin\theta + vt_0)}$$
$$-\left(i2\pi\hat{k}\right) H_x^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$+\left(i2\pi\hat{k}\right) H_x^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\sin\theta\right)^2 H_x^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\sin\theta\right)^2 H_x^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\cos\theta\right)^2 H_y^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\cos\theta\right)^2 H_y^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} \hat{x}\, d\hat{y}, \tag{2.147}$$

and

$$P_3(x,y,t,t_0) = H_y^+ e^{i\hat{k}(x\cos\theta + y\sin\theta - vt_0)} + H_y^- e^{i\hat{k}(x\cos\theta + y\sin\theta + vt_0)}$$
$$-\left(i2\pi\hat{k}\right) H_x^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$+\left(i2\pi\hat{k}\right) H_x^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\sin\theta\right)^2 H_y^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$+\left(2\pi\hat{k}\sin\theta\right)^2 H_y^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\cos\theta\right)^2 H_x^+ e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta - vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}$$
$$-\left(2\pi\hat{k}\cos\theta\right)^2 H_x^- e^{i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta + vt_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(\hat{x},\hat{y},t-t_0) e^{-i2\pi\hat{k}(\hat{x}\cos\theta + \hat{y}\sin\theta)} d\hat{x}\, d\hat{y}. \tag{2.148}$$

The integrals involving $F_1(x,y,t)$ and $F_2(x,y,t)$ in equations (2.146) to (2.148) are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1\left(\hat{x},\hat{y},t-t_0\right)e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} = \frac{1}{(2\pi)^2}\left(\frac{1-\cos\left(2\pi\hat{k}vt\right)}{\hat{k}^2}\right) \tag{2.149}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2\left(\hat{x},\hat{y},t-t_0\right)e^{-i2\pi\hat{k}(\hat{x}\cos\theta+\hat{y}\sin\theta)}d\hat{x}\,d\hat{y} = \frac{1}{2\pi}\left(\frac{\sin\left(2\pi\hat{k}vt\right)}{\hat{k}}\right). \tag{2.150}$$

Substituting expressions (2.149) and (2.150) into equations (2.146) to (2.148) and simplifying the expressions yields

$$\begin{bmatrix} P_1\left(x,y,t,t_0\right) \\ P_2\left(x,y,t,t_0\right) \\ P_3\left(x,y,t,t_0\right) \end{bmatrix} = \begin{bmatrix} E_z^+ \\ H_x^+ \\ H_y^+ \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta-vt)} + \begin{bmatrix} E_z^- \\ H_x^- \\ H_y^- \end{bmatrix} e^{i2\pi\hat{k}(x\cos\theta+y\sin\theta+vt)} = \begin{bmatrix} E_z\left(x,y,t\right) \\ H_x\left(x,y,t\right) \\ H_y\left(x,y,t\right) \end{bmatrix}. \tag{2.151}$$

which is exactly the plane wave at time t. This proves that the direct application of equations (2.118) to (2.120) to a two-dimensional plane wave at time t_0 correctly yields the two-dimensional plane wave at time $t > t_0$.

2.3 Three-Dimensional Equation

In this section the solution to the three-dimensional form of Maxwell's equations is derived. In this case the relevant components are $E_x(x,y,z,t)$, $E_y(x,y,z,t)$, $E_z(x,y,z,t)$, $H_x(x,y,z,t)$, $H_y(x,y,z,t)$, and $H_z(x,y,z,t)$. The variables $F_1 = -\sigma_e(x,y,z)/\varepsilon(x,y,z)$, $F_2 = -\sigma_m(x,y,z)/\mu(x,y,z)$, $F_3 = 1/\varepsilon(x,y,z)$, and $F_4 = 1/\mu(x,y,z)$ are used to simplify the notation. The wave propagates according to

$$\frac{\partial}{\partial t}\mathbf{g}(x,y,z,t) = \hat{\mathbf{P}}(x,y,z)\mathbf{g}(x,y,z,t) + \mathbf{r}(x,y,z,t), \tag{2.152}$$

where

$$\mathbf{g}(x,y,z,t) = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix}, \tag{2.153}$$

$$\hat{\mathbf{P}}(x,y,z) = \begin{bmatrix} F_1 & 0 & 0 & 0 & -F_3\frac{\partial}{\partial z} & F_3\frac{\partial}{\partial y} \\ 0 & F_1 & 0 & F_3\frac{\partial}{\partial z} & 0 & -F_3\frac{\partial}{\partial x} \\ 0 & 0 & F_1 & -F_3\frac{\partial}{\partial y} & F_3\frac{\partial}{\partial x} & 0 \\ 0 & F_4\frac{\partial}{\partial z} & -F_4\frac{\partial}{\partial y} & F_2 & 0 & 0 \\ -F_4\frac{\partial}{\partial z} & 0 & F_4\frac{\partial}{\partial x} & 0 & F_2 & 0 \\ F_4\frac{\partial}{\partial y} & -F_4\frac{\partial}{\partial x} & 0 & 0 & 0 & F_2 \end{bmatrix}, \tag{2.154}$$

and

$$\mathbf{r}(x,y,z,t)=\begin{bmatrix} -F_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -F_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -F_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -F_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -F_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_4 \end{bmatrix}\begin{bmatrix} J_x(x,y,z,t) \\ J_y(x,y,z,t) \\ J_z(x,y,z,t) \\ M_x(x,y,z,t) \\ M_y(x,y,z,t) \\ M_z(x,y,z,t) \end{bmatrix}. \tag{2.155}$$

Taking the Fourier transform from the (x,y,z) domain to the (p,q,r) domain on both sides of the equation yields

$$\begin{aligned} \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}(x,y,z,t)e^{-i2\pi(px+qy)}dxdydz = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\mathbf{P}}(x,y,z)\mathbf{g}(x,y,z,t)e^{-i2\pi(px+qy+rz)}dxdydz \\ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{r}(x,y,z,t)e^{-i2\pi(px+qy+rz)}dxdydz. \end{aligned} \tag{2.156}$$

Using the following properties of the Fourier transform

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}m(x,y,z)n(x,y,z)e^{-i2\pi(px+qy+rz)}dxdydz = \\ M(p,q,r)*N(p,q,r)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}M(p-\hat{p},q-\hat{q},r-\hat{r})N(\hat{p},\hat{q},\hat{r})\,d\hat{p}\,d\hat{q}\,d\hat{r}, \end{aligned} \tag{2.157}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}m(x,y,z)\right\}e^{-i2\pi(px+qy+rz)}dxdydz = i2\pi pM(p,q,r), \tag{2.158}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial y}m(x,y,z)\right\}e^{-i2\pi(px+qy+rz)}dxdydz = i2\pi qM(p,q,r), \tag{2.159}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial z}m(x,y,z)\right\}e^{-i2\pi(px+qy+rz)}dx\,dy\,dz = i2\pi r M(p,q,r) \tag{2.160}$$

equation (2.156) can be written as the following six equations

$$\begin{aligned}\frac{\partial}{\partial t}E_x(p,q,r,t) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(p-\hat{p},q-\hat{q},r-\hat{r})E_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi rF_3(p-\hat{p},q-\hat{q},r-\hat{r})H_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_3(p-\hat{p},q-\hat{q},r-\hat{r})H_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_3(p-\hat{p},q-\hat{q},r-\hat{r})J_x(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau,\end{aligned} \tag{2.161}$$

$$\begin{aligned}\frac{\partial}{\partial t}E_y(p,q,r,t) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(p-\hat{p},q-\hat{q},r-\hat{r})E_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{r}F_3(p-\hat{p},q-\hat{q},r-\hat{r})H_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_3(p-\hat{p},q-\hat{q},r-\hat{r})H_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_3(p-\hat{p},q-\hat{q},r-\hat{r})J_y(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau,\end{aligned} \tag{2.162}$$

$$\begin{aligned}\frac{\partial}{\partial t}E_z(p,q,r,t) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(p-\hat{p},q-\hat{q},r-\hat{r})E_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_3(p-\hat{p},q-\hat{q},r-\hat{r})H_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_3(p-\hat{p},q-\hat{q},r-\hat{r})H_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\\ &-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_3(p-\hat{p},q-\hat{q},r-\hat{r})J_z(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau,\end{aligned} \tag{2.163}$$

$$\frac{\partial}{\partial t}H_x(p,q,r,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(p-\hat{p},q-\hat{q},r-\hat{r})H_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{r}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\tag{2.164}$$
$$-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_4(p-\hat{p},q-\hat{q},r-\hat{r})M_x(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau,$$

$$\frac{\partial}{\partial t}H_y(p,q,r,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(p-\hat{p},q-\hat{q},r-\hat{r})H_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{r}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\tag{2.165}$$
$$-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_4(p-\hat{p},q-\hat{q},r-\hat{r})M_y(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau,$$

$$\frac{\partial}{\partial t}H_z(p,q,r,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(p-\hat{p},q-\hat{q},r-\hat{r})H_z(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{q}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_x(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}i2\pi\hat{p}F_4(p-\hat{p},q-\hat{q},r-\hat{r})E_y(\hat{p},\hat{q},\hat{r},t)\,d\hat{p}\,d\hat{q}\,d\hat{r}\tag{2.166}$$
$$-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_4(p-\hat{p},q-\hat{q},r-\hat{r})M_z(\hat{p},\hat{q},\hat{r},\tau)\,d\hat{p}\,d\hat{q}\,d\hat{r}\,d\tau.$$

Equations (2.161) to (2.166) are integro-differential equations. Obtaining general analytic solutions to these equations is extremely difficult, if possible at all. On the other hand, if the medium is homogeneous a solution can indeed be obtained as it is shown in the next sections.

2.3.1 Solution to the Three-Dimensional Equation in a Homogeneous Medium

In the special case of a *homogeneous* medium where the parameters ε, μ, σ, and ρ are constants, $F_1(p,q,r)=(-\sigma_e/\varepsilon)\delta(p,q,r)$, $F_2(p,q,r)=(-\sigma_m/\mu)\delta(p,q,r)$, $F_3(p,q,r)=(1/\varepsilon)\delta(p,q,r)$, and $F_4(p,q,r)=(1/\mu)\delta(p,q,r)$. In this case equations (2.161) to (2.166) become

$$\frac{\partial}{\partial t}\mathbf{G}(p,q,r,t)=\mathbf{P}(p,q,r)\mathbf{G}(p,q,r,t)+\mathbf{R}(p,q,r,t), \qquad (2.167)$$

where

$$\mathbf{P}(p,q,r)=\begin{bmatrix} -\frac{\sigma_e}{\varepsilon} & 0 & 0 & 0 & -\frac{i2\pi r}{\varepsilon} & \frac{i2\pi q}{\varepsilon} \\ 0 & -\frac{\sigma_e}{\varepsilon} & 0 & \frac{i2\pi r}{\varepsilon} & 0 & -\frac{i2\pi p}{\varepsilon} \\ 0 & 0 & -\frac{\sigma_e}{\varepsilon} & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} & 0 \\ 0 & \frac{i2\pi r}{\mu} & -\frac{i2\pi q}{\mu} & -\frac{\sigma_m}{\mu} & 0 & 0 \\ -\frac{i2\pi r}{\mu} & 0 & \frac{i2\pi p}{\mu} & 0 & -\frac{\sigma_m}{\mu} & 0 \\ \frac{i2\pi q}{\mu} & -\frac{i2\pi p}{\mu} & 0 & 0 & 0 & -\frac{\sigma_m}{\mu} \end{bmatrix}, \qquad (2.168)$$

$$\mathbf{G}(p,q,r,t)=\begin{bmatrix} E_x(p,q,r,t) \\ E_y(p,q,r,t) \\ E_z(p,q,r,t) \\ H_x(p,q,r,t) \\ H_y(p,q,r,t) \\ H_z(p,q,r,t) \end{bmatrix}, \tag{2.169}$$

and

$$\mathbf{R}(p,q,r,t)=-\begin{bmatrix} J_x(p,q,r,t)/\varepsilon \\ J_y(p,q,r,t)/\varepsilon \\ J_z(p,q,r,t)/\varepsilon \\ M_x(p,q,r,t)/\mu \\ M_y(p,q,r,t)/\mu \\ M_z(p,q,r,t)/\mu \end{bmatrix}. \tag{2.170}$$

The solution to equation (2.167) at time t is the three-dimensional version of equation (2.13) given by

$$\mathbf{G}(p,q,r,t)=\mathbf{H}(p,q,r,t-t_0)\mathbf{G}(p,q,r,t_0)+\int_{t_0}^{t}\mathbf{H}(p,q,r,t-\tau)\mathbf{R}(p,q,r,\tau)d\tau, \tag{2.171}$$

where $\mathbf{G}(p,q,r,t_0)$ is the solution at time t_0.

In the special case of a lossless two-dimensional homogeneous medium where $\sigma_e=0$, $\sigma_m=0$, and ε and μ are constants we have

$$
\mathbf{P}(p,q,r)=\begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{i2\pi r}{\varepsilon} & \frac{i2\pi q}{\varepsilon} \\
0 & 0 & 0 & \frac{i2\pi r}{\varepsilon} & 0 & -\frac{i2\pi p}{\varepsilon} \\
0 & 0 & 0 & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} & 0 \\
0 & \frac{i2\pi r}{\mu} & -\frac{i2\pi q}{\mu} & 0 & 0 & 0 \\
-\frac{i2\pi r}{\mu} & 0 & \frac{i2\pi p}{\mu} & 0 & 0 & 0 \\
\frac{i2\pi q}{\mu} & -\frac{i2\pi p}{\mu} & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{2.172}
$$

The kernel matrix $\mathbf{H}(p,q,r,t)$ in this case (see appendix B) is

$$
\mathbf{H}(p,q,r,t)=
\begin{bmatrix}
1-(q^2+r^2)F & pqF & prF & 0 & -ir\eta G & iq\eta G \\
pqF & 1-(p^2+r^2)F & qrF & ir\eta G & 0 & -ip\eta G \\
prF & qrF & 1-(p^2+q^2)F & -iq\eta G & ip\eta G & 0 \\
0 & irG/\eta & -iqG/\eta & 1-(q^2+r^2)F & pqF & prF \\
-irG/\eta & 0 & ipG/\eta & pqF & 1-(p^2+r^2)F & qrF \\
iqG/\eta & -ipG/\eta & 0 & prF & qrF & 1-(p^2+q^2)F
\end{bmatrix}, \tag{2.173}
$$

where $F=\left(1-\cos(2\pi kvt)\right)/k^2$, $G=\sin(2\pi kvt)/k$, $k=\sqrt{p^2+q^2+r^2}$, $v=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.

Substituting the kernel matrix $\mathbf{H}(p,q,r,t)$ from (2.173) into equation (2.171) and then applying the inverse Fourier transform produces the following six equations

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(p,q,r,t)e^{i2\pi(px+qy+rz)}dp\,dq\,dr = \\
&\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)\left((i2\pi q)^2+(i2\pi r)^2\right)E_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi q)E_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi r)E_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi r)\eta H_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi q)\eta H_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}J_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)\left((i2\pi q)^2+(i2\pi r)^2\right)\frac{1}{\varepsilon}J_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi q)\frac{1}{\varepsilon}J_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi r)\frac{1}{\varepsilon}J_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi r)\frac{\eta}{\mu}M_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi q)\frac{\eta}{\mu}M_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau,
\end{aligned}
$$

(2.174)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(p,q,r,t)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr =$$

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)(i2\pi p)(i2\pi q)\, E_x(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi r)^2\right) E_y(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)(i2\pi q)(i2\pi r)\, E_z(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\right)(i2\pi r)\,\eta H_x(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\right)(i2\pi p)\,\eta H_z(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr$$

$$+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-\tau\right)\right)}{k^2}\right)(i2\pi p)(i2\pi q)\frac{1}{\varepsilon} J_x(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau$$

$$-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varepsilon} J_y(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau$$

$$-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-\tau\right)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi r)^2\right)\frac{1}{\varepsilon} J_y(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau$$

$$+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-\tau\right)\right)}{k^2}\right)(i2\pi q)(i2\pi r)\frac{1}{\varepsilon} J_z(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau$$

$$-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-\tau\right)\right)}{k}\right)(i2\pi r)\frac{\eta}{\mu} M_x(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau$$

$$+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-\tau\right)\right)}{k}\right)(i2\pi p)\frac{\eta}{\mu} M_z(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau, \tag{2.175}$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_z(p,q,r,t)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr = \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-t_0)\right)}{k^2}\right)(i2\pi p)(i2\pi r)\, E_x(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-t_0)\right)}{k^2}\right)(i2\pi q)(i2\pi r)\, E_y(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_z(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-t_0)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi q)^2\right) E_z(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\right)(i2\pi q)\,\eta H_x(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\right)(i2\pi p)\,\eta H_y(p,q,r,t_0)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-\tau)\right)}{k^2}\right)(i2\pi p)(i2\pi r)\frac{1}{\varepsilon} J_x(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-\tau)\right)}{k^2}\right)(i2\pi q)(i2\pi r)\frac{1}{\varepsilon} J_y(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\varepsilon} J_z(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv(t-\tau)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi q)^2\right)\frac{1}{\varepsilon} J_z(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv(t-\tau)\right)}{k}\right)(i2\pi q)\frac{\eta}{\mu} M_x(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv(t-\tau)\right)}{k}\right)(i2\pi p)\frac{\eta}{\mu} M_y(p,q,r,\tau)\, e^{i2\pi(px+qy+rz)} dp\, dq\, dr\, d\tau,
\end{aligned}
$$

(2.176)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_x(p,q,r,t)e^{i2\pi(px+qy+rz)}dp\,dq\,dr =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi r)\frac{1}{\eta}E_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi q)\frac{1}{\eta}E_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)\left((i2\pi q)^2+(i2\pi r)^2\right)H_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi q)H_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi r)H_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$

$$-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi r)\frac{1}{\eta\varepsilon}J_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau$$

$$+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi q)\frac{1}{\eta\varepsilon}J_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau$$

$$-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu}M_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau$$

$$-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)\left((i2\pi q)^2+(i2\pi r)^2\right)\frac{1}{\mu}M_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau$$

$$+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi q)\frac{1}{\mu}M_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau$$

$$+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi r)\frac{1}{\mu}M_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau,$$

(2.177)

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_y(p,q,r,t)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr = \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi r)\frac{1}{\eta}E_x(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-t_0))}{k}\right)(i2\pi p)\frac{1}{\eta}E_z(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi p)(i2\pi q)H_x(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_y(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)\left((i2\pi p)^2+(i2\pi r)^2\right)H_y(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)(i2\pi q)(i2\pi r)H_z(p,q,r,t_0)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi r)\frac{1}{\eta\varepsilon}J_x(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kv(t-\tau))}{k}\right)(i2\pi p)\frac{1}{\eta\varepsilon}J_z(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi p)(i2\pi q)\frac{1}{\mu}M_x(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu}M_y(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-t_0))}{k^2}\right)\left((i2\pi p)^2+(i2\pi r)^2\right)\frac{1}{\mu}M_y(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kv(t-\tau))}{k^2}\right)(i2\pi q)(i2\pi r)\frac{1}{\mu}M_z(p,q,r,\tau)\,e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau,
\end{aligned}
\tag{2.178}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_z(p,q,r,t)e^{i2\pi(px+qy+rz)}dp\,dq\,dr = \\
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\right)(i2\pi q)\frac{1}{\eta}E_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\right)(i2\pi p)\frac{1}{\eta}E_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)(i2\pi p)(i2\pi r)H_x(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)(i2\pi q)(i2\pi r)H_y(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi q)^2\right)H_z(p,q,r,t_0)e^{i2\pi(px+qy+rz)}dp\,dq\,dr \\
&-\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-\tau\right)\right)}{k}\right)(i2\pi q)\frac{1}{\eta\varepsilon}J_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{2\pi}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin\left(2\pi kv\left(t-\tau\right)\right)}{k}\right)(i2\pi p)\frac{1}{\eta\varepsilon}J_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-\tau\right)\right)}{k^2}\right)(i2\pi p)(i2\pi r)\frac{1}{\mu}M_x(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&+\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-t_0\right)\tau\right)}{k^2}\right)(i2\pi q)(i2\pi r)\frac{1}{\mu}M_y(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu}M_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau \\
&-\frac{1}{(2\pi)^2}\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos\left(2\pi kv\left(t-\tau\right)\right)}{k^2}\right)\left((i2\pi p)^2+(i2\pi q)^2\right)\frac{1}{\mu}M_z(p,q,r,\tau)e^{i2\pi(px+qy+rz)}dp\,dq\,dr\,d\tau.
\end{aligned}
$$

(2.179)

Using the differentiation properties of the Fourier transform, equations (2.174) to (2.179) can be written as

$$
\begin{aligned}
&E_x(x,y,z,t) = E_x(x,y,z,t_0) + F_1(x,y,z,t-t_0) * \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x(x,y,z,t_0) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial y} E_y(x,y,z,t_0) - F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial z} E_z(x,y,z,t_0) \\
&-\eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial z} H_y(x,y,z,t_0) + \eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial y} H_z(x,y,z,t_0) \\
&-\int_{t_0}^{t} \frac{1}{\varepsilon} J_x(x,y,z,\tau)\, d\tau - \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\varepsilon} J_x(x,y,z,\tau) \right] d\tau \\
&+\int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial y} \frac{1}{\varepsilon} J_y(x,y,z,\tau) \right] d\tau \\
&+\int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial z} \frac{1}{\varepsilon} J_z(x,y,z,\tau) \right] d\tau \\
&+\int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial z} \frac{1}{\mu} M_y(x,y,z,\tau) \right] d\tau \\
&-\int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial y} \frac{1}{\mu} M_z(x,y,z,\tau) \right] d\tau, \qquad (2.180)
\end{aligned}
$$

$$
\begin{aligned}
E_y(x,y,z,t) &= -F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial y} E_x(x,y,z,t_0) \\
&+ E_y(x,y,z,t_0) + F_1(x,y,z,t-t_0) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) E_y(x,y,z,t_0) \\
&- F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial y \partial z} E_z(x,y,z,t_0) \\
&+ \eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial z} H_x(x,y,z,t_0) - \eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial x} H_z(x,y,z,t_0) \\
&+ \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial y} \frac{1}{\varepsilon} J_x(x,y,z,\tau) \right] d\tau \\
&- \int_{t_0}^{t} \frac{1}{\varepsilon} J_y(x,y,z,\tau) \, d\tau - \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \frac{1}{\varepsilon} J_y(x,y,z,\tau) \right] d\tau \\
&+ \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial y \partial z} \frac{1}{\varepsilon} J_z(x,y,z,\tau) \right] d\tau \\
&- \int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial z} \frac{1}{\mu} M_x(x,y,z,\tau) \right] d\tau \\
&+ \int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial x} \frac{1}{\mu} M_z(x,y,z,\tau) \right] d\tau,
\end{aligned}
\tag{2.181}
$$

$$
\begin{aligned}
E_z(x,y,z,t) &= -F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial z} E_x(x,y,z,t_0) \\
&- F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial y \partial z} E_y(x,y,z,t_0) \\
&+ E_z(x,y,z,t_0) + F_1(x,y,z,t-t_0) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_z(x,y,z,t_0) \\
&- \eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial y} H_x(x,y,z,t_0) + \eta F_2(x,y,z,t-t_0) * \frac{\partial}{\partial x} H_y(x,y,z,t_0) \\
&+ \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial z} \frac{1}{\varepsilon} J_x(x,y,z,\tau) \right] d\tau \\
&+ \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial y \partial z} \frac{1}{\varepsilon} J_y(x,y,z,\tau) \right] d\tau \\
&- \int_{t_0}^{t} \frac{1}{\varepsilon} J_z(x,y,z,\tau) d\tau - \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{1}{\varepsilon} J_z(x,y,z,\tau) \right] d\tau \\
&+ \int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial y} \frac{1}{\mu} M_x(x,y,z,\tau) \right] d\tau \\
&- \int_{t_0}^{t} \left[\eta F_2(x,y,z,t-\tau) * \frac{\partial}{\partial x} \frac{1}{\mu} M_y(x,y,z,\tau) \right] d\tau,
\end{aligned} \tag{2.182}
$$

$$
\begin{aligned}
&H_x(x,y,z,t) = F_2(x,y,z,t-t_0) * \frac{\partial}{\partial z}\frac{1}{\eta} E_y(x,y,z,t_0) \\
&-F_2(x,y,z,t-t_0) * \frac{\partial}{\partial y}\frac{1}{\eta} E_z(x,y,z,t_0) \\
&+H_x(x,y,z,\tau) + F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) H_x(x,y,z,\tau) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial y} H_y(x,y,z,t_0) - F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial z} H_z(x,y,z,t_0) \\
&-\int_{t_0}^{t}\left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial z}\frac{1}{\eta\varepsilon} J_y(x,y,z,\tau)\right] d\tau \\
&+\int_{t_0}^{t}\left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial y}\frac{1}{\eta\varepsilon} J_z(x,y,z,\tau)\right] d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu} M_x(x,y,z,\tau)\, d\tau - \int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{\mu} M_x(x,y,z,\tau)\right] d\tau \\
&+\int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial y}\frac{1}{\mu} M_y(x,y,z,\tau)\right] d\tau \\
&+\int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial z}\frac{1}{\mu} M_z(x,y,z,\tau)\right] d\tau,
\end{aligned}
\tag{2.183}
$$

$$
\begin{aligned}
&H_y(x,y,z,t) = -F_2(x,y,z,t-t_0) * \frac{\partial}{\partial z}\frac{1}{\eta} E_x(x,y,z,t_0) \\
&+F_2(x,y,z,t-t_0) * \frac{\partial}{\partial x}\frac{1}{\eta} E_z(x,y,z,t_0) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial y} H_x(x,y,z,t_0) \\
&+H_y(x,y,z,t_0) + F_1(x,y,z,t-t_0) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) H_y(x,y,z,t_0) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial y} H_z(x,y,z,t_0) \\
&+\int_{t_0}^{t} \left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial z}\frac{1}{\eta\varepsilon} J_x(x,y,z,\tau) \right] d\tau \\
&-\int_{t_0}^{t} \left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial x}\frac{1}{\eta\varepsilon} J_z(x,y,z,\tau) \right] d\tau \\
&+\int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial y}\frac{1}{\mu} M_x(x,y,z,\tau) \right] d\tau \\
&-\int_{t_0}^{t} \frac{1}{\mu} M_y(x,y,z,\tau)\, d\tau - \int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\mu} M_y(x,y,z,\tau) \right] d\tau \qquad (2.184) \\
&+\int_{t_0}^{t} \left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial y}\frac{1}{\mu} M_z(x,y,z,\tau) \right] d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&H_z(x,y,z,t) = F_2(x,y,z,t-t_0) * \frac{\partial}{\partial y}\frac{1}{\eta}E_x(x,y,z,t_0) \\
&+F_2(x,y,z,t-t_0) * \frac{\partial}{\partial x}\frac{1}{\eta}E_y(x,y,z,t_0) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial x \partial z}H_x(x,y,z,t_0) \\
&-F_1(x,y,z,t-t_0) * \frac{\partial^2}{\partial y \partial z}H_y(x,y,z,t_0) \\
&+H_z(x,y,z,t_0) + F_1(x,y,z,t-t_0) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)H_z(x,y,z,t_0) \\
&-\int_{t_0}^{t}\left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial y}\frac{1}{\eta\varepsilon}J_x(x,y,z,\tau)\right]d\tau \\
&-\int_{t_0}^{t}\left[F_2(x,y,z,t-\tau) * \frac{\partial}{\partial x}\frac{1}{\eta\varepsilon}J_y(x,y,z,\tau)\right]d\tau \\
&+\int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial x \partial z}\frac{1}{\mu}M_x(x,y,z,\tau)\right]d\tau \\
&+\int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \frac{\partial^2}{\partial y \partial z}\frac{1}{\mu}M_y(x,y,z,\tau)\right]d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_z(x,y,z,\tau)\,d\tau - \int_{t_0}^{t}\left[F_1(x,y,z,t-\tau) * \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{\mu}M_z(x,y,z,\tau)\right]d\tau,
\end{aligned}
\tag{2.185}
$$

where $*$ denotes convolution and

$$
\begin{aligned}
F_1(x,y,z,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,q,r,t)\,e^{i2\pi(px+qy+rz)}\,dp\,dq\,dr \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kvt)}{k^2}\right)e^{i2\pi(px+qy+rz)}\,dp\,dq\,dr
\end{aligned}
\tag{2.186}
$$

and

$$F_2(x,y,z,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(p,q,r,t)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\sin(2\pi kvt)}{k}\right)e^{i2\pi(px+qy+rz)}dp\,dq\,dr. \tag{2.187}$$

To obtain closed-form expressions for (2.186) and (2.187), it is advantageous to exploit the inherent symmetry of the functions. When $M(p,q,r)$ is a function of $k=\sqrt{p^2+q^2+r^2}$, that is when $M(p,q,r)=M(k)$, its inverse Fourier transform $m(x,y,z)$ is a function of $s=\sqrt{x^2+y^2+z^2}$. In other words, $m(x,y,z)=m(s)$. Similarly, if $m(x,y,z)$ is a function of $s=\sqrt{x^2+y^2+z^2}$, then $M(p,q,r)$ is a function of $k=\sqrt{p^2+q^2+r^2}$. In this case, we can use [2]

$$M(p,q,r)=M(k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}m(x,y,z)e^{-i2\pi(px+qy+rz)}dx\,dy\,dz$$
$$=4\pi\int_0^{\infty}m(s)\operatorname{sinc}(2ks)s^2ds \tag{2.188}$$

and

$$m(x,y,z)=m(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(p,q,r)e^{i2\pi(px+qy+rz)}dp\,dq\,dr$$
$$=4\pi\int_0^{\infty}M(k)\operatorname{sinc}(2ks)k^2dk \tag{2.189}$$

Since $F(p,q,r,t)$ and $G(p,q,r,t)$ are functions of $k=\sqrt{p^2+q^2+r^2}$, it is the case that $F_1(x,y,z,t)$ and $F_2(x,y,z,t)$ are functions of $s=\sqrt{x^2+y^2+z^2}$. Using identities (2.188) and (2.189), equation (2.187) becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(2\pi kvt)}{k}e^{i2\pi(px+qy+rz)}dp\,dq\,dr = 4\pi\int_{0}^{\infty}\frac{\sin(2\pi kvt)}{k}\operatorname{sinc}(2ks)k^2dk\\
&=\frac{2}{s}\int_{0}^{\infty}\sin(2\pi kvt)\sin(2\pi ks)\,dk\\
&=\frac{1}{s}\int_{-\infty}^{\infty}\left(\frac{e^{i2\pi kvt}-e^{-i2\pi kvt}}{2i}\right)\left(\frac{e^{i2\pi ks}-e^{-i2\pi ks}}{2i}\right)dk\\
&=\frac{1}{4s}\left[-\int_{-\infty}^{\infty}e^{i2\pi k(vt+s)}ds+\int_{-\infty}^{\infty}e^{i2\pi k(vt-s)}ds+\int_{-\infty}^{\infty}e^{-i2\pi k(vt-s)}ds-\int_{-\infty}^{\infty}e^{-i2\pi k(vt+s)}ds\right]\\
&=\frac{\delta(vt-s)}{2s}-\frac{\delta(vt+s)}{2s}.
\end{aligned}
\tag{2.190}
$$

In order to check whether this solution is valid let us consider the convolution of a function $g(x,y,z)$ with the kernel $\delta(vt-s)/s$.

$$
z(x,y,z,vt)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(vt-r)}{r}g(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0\,, \tag{2.191}
$$

where $r^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$. Changing to spherical polar angular coordinates θ,ϕ using (x,y,z) as the origin, equation (2.191) can be written as

$$
\begin{aligned}
&z(x,y,z,vt)\\
&=\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{\infty}\frac{\delta(vt-R)}{R}g(x+R\sin\theta\cos\phi,y+R\sin\theta\sin\phi,z+R\cos\theta)R^2\sin\theta\,dR\,d\theta\,d\phi \quad (2.192)\\
&=vt\int_{0}^{\pi}\int_{0}^{2\pi}g(x+vt\sin\phi\cos\theta,y+vt\sin\phi\sin\theta,z+vt\cos\phi)\sin\theta\,d\theta\,d\phi.
\end{aligned}
$$

The integral is computed over the sphere of center (x, y, z) and radius a. Because the radius of the sphere must be positive in order to have a physical meaning the second component of equation (2.190) is rejected. Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(2\pi kvt)}{k}e^{i2\pi(px+qy+rz)}dp\,dq\,dr=\frac{\delta(vt-s)}{2s}-\cancel{\frac{\delta(vt+s)}{2s}}=\frac{\delta(vt-s)}{2s}. \qquad (2.193)$$

To obtain a closed-form expression to (2.186) the following identity is used:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(2\pi kvt)}{k}e^{i2\pi(px+qy+rz)}dp\,dq\,dr=\\&\qquad\frac{1}{2\pi v}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kvt)}{k^2}\right)e^{i2\pi(px+qy+rz)}dp\,dq\,dr.\end{aligned} \qquad (2.194)$$

Combining this identity and equation (2.193) and simplifying terms yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1-\cos(2\pi kvt)}{k^2}\right)e^{i2\pi(px+qy+rz)}dp\,dq\,dr=\pi\frac{u(vt-s)}{s}, \qquad (2.195)$$

where $u(x)$ is the well-known unit step function.

Equations (2.180) to (2.185) can now be written as

$$
\begin{aligned}
&E_x(x,y,z,t) = E_x(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{y}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)E_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}E_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}E_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\,\eta\frac{\partial}{\partial\hat{z}}H_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\,\eta\frac{\partial}{\partial\hat{y}}H_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{t_0}^{t}\frac{1}{\varepsilon}J_x(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\hat{y}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)J_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}J_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}J_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \qquad (2.196) \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{z}}M_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{y}}M_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
&E_y(x,y,z,t)= \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}E_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+E_y(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)E_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}E_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\eta\frac{\partial}{\partial\hat{z}}H_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\eta\frac{\partial}{\partial\hat{x}}H_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}J_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\frac{1}{\varepsilon}J_y(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)J_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}J_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{z}}M_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{x}}M_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau,
\end{aligned}
\tag{2.197}
$$

$$
\begin{aligned}
&E_z(x,y,z,t)= \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}E_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}E_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+E_z(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{y}^2}\right)E_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\eta\frac{\partial}{\partial\hat{y}}H_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\eta\frac{\partial}{\partial\hat{x}}H_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}J_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}J_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\varepsilon}J_z(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\varepsilon}\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{y}^2}\right)J_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{y}}M_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{\eta}{\mu}\frac{\partial}{\partial\hat{x}}M_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau,
\end{aligned}
\tag{2.198}
$$

$$
\begin{aligned}
&H_x(x,y,z,t) = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{z}}E_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{y}}E_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+H_x(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{y}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)H_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}H_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}H_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{z}}J_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{y}}J_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_x(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\left(\frac{\partial^2}{\partial\hat{y}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)M_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}M_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}M_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau,
\end{aligned}
\tag{2.199}
$$

$$
\begin{aligned}
&H_y(x,y,z,t)= \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{z}}E_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial(x-\alpha)}E_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial(x-\alpha)\partial(y-\beta)}H_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+H_y(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)H_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}H_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{z}}J_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{x}}J_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}M_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \qquad (2.200) \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_y(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)M_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{y}}M_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&H_z(x,y,z,t) = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{y}}E_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{1}{\eta}\frac{\partial}{\partial\hat{x}}E_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}H_x(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}H_y(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&+H_z(x,y,z,t_0) \\
&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-t_0)\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)H_z(\hat{x},\hat{y},\hat{z},t_0)\,d\hat{x}\,d\hat{y}\,d\hat{z} \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{y}}J_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\eta\varepsilon}\frac{\partial}{\partial\hat{x}}J_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{x}\partial\hat{z}}M_x(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&+\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\frac{\partial^2}{\partial\hat{y}\partial\hat{z}}M_y(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau \\
&-\int_{t_0}^{t}\frac{1}{\mu}M_z(x,y,z,\tau)\,d\tau \\
&-\int_{t_0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x-\hat{x},y-\hat{y},z-\hat{z},t-\tau)\frac{1}{\mu}\left(\frac{\partial^2}{\partial\hat{x}^2}+\frac{\partial^2}{\partial\hat{z}^2}\right)M_z(\hat{x},\hat{y},\hat{z},\tau)\,d\hat{x}\,d\hat{y}\,d\hat{z}\,d\tau.
\end{aligned}
\tag{2.201}
$$

where $F_1 = \pi u(vt-s)/s$, $F_2 = \delta(vt-s)/2s$, and $s = \sqrt{x^2+y^2+z^2}$.

Expressions (2.196) to (2.201) are exact analytic solutions to the three-dimensional form of Maxwell's equations in a lossless homogeneous medium. Given times t_0 and t, where $t > t_0$, if the electric and magnetic fields are known at t_0 and the current and magnetic sources are known from t_0 to t, these equations can be used to compute the electric and magnetic fields at time t.

2.3.2 Time Evolution of Three-Dimensional Plane Waves

In this section, it is shown that in a source-free lossless homogeneous medium, application of the kernel matrix $\mathbf{H}(p,q,r,t-t_0)$, given in (2.173) to a plane wave at time t_0, results in the plane wave at time t. In a homogeneous medium with $(\mathbf{J}_{source} = \mathbf{M}_{source} = 0)$ and $\sigma_e = \sigma_m = 0$, the three-dimensional form of Maxwell's curl equations is

$$\frac{\partial}{\partial t} E_x(x,y,z,t) = \frac{1}{\varepsilon}\frac{\partial}{\partial y} H_z(x,y,z,t) - \frac{1}{\varepsilon}\frac{\partial}{\partial z} H_y(x,y,z,t), \tag{2.202}$$

$$\frac{\partial}{\partial t} E_y(x,y,z,t) = \frac{1}{\varepsilon}\frac{\partial}{\partial z} H_x(x,y,z,t) - \frac{1}{\varepsilon}\frac{\partial}{\partial x} H_z(x,y,z,t), \tag{2.203}$$

$$\frac{\partial}{\partial t} E_z(x,y,z,t) = \frac{1}{\varepsilon}\frac{\partial}{\partial x} H_y(x,y,z,t) - \frac{1}{\varepsilon}\frac{\partial}{\partial y} H_x(x,y,z,t), \tag{2.204}$$

$$\frac{\partial}{\partial t} H_x(x,y,z,t) = \frac{1}{\mu}\frac{\partial}{\partial z} E_y(x,y,z,t) - \frac{1}{\mu}\frac{\partial}{\partial y} E_z(x,y,z,t), \tag{2.205}$$

$$\frac{\partial}{\partial t} H_y(x,y,z,t) = \frac{1}{\mu}\frac{\partial}{\partial x} E_z(x,y,z,t) - \frac{1}{\mu}\frac{\partial}{\partial z} E_x(x,y,z,t), \tag{2.206}$$

and

$$\frac{\partial}{\partial t}H_z(x,y,z,t)=\frac{1}{\mu}\frac{\partial}{\partial y}E_x(x,y,z,t)-\frac{1}{\mu}\frac{\partial}{\partial x}E_y(x,y,z,t). \tag{2.207}$$

A three-dimensional plane wave with wavenumber $\hat{k}$ is given by

$$\begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix} = \left\{ \begin{bmatrix} E_{x0}^{+} \\ E_{y0}^{+} \\ E_{z0}^{+} \\ H_{x0}^{+} \\ H_{y0}^{+} \\ H_{z0}^{+} \end{bmatrix} e^{-i2\pi\hat{k}vt} + \begin{bmatrix} E_{x0}^{-} \\ E_{y0}^{-} \\ E_{z0}^{-} \\ H_{x0}^{-} \\ H_{y0}^{-} \\ H_{z0}^{-} \end{bmatrix} e^{i2\pi\hat{k}vt} \right\} e^{i2\pi\hat{\kappa}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)}. \tag{2.208}$$

Substituting (2.208) into equations (2.202) to (2.207) and equating the common terms yields

$$\begin{aligned} E_{x0}^{+} &= -\eta\sin\theta\sin\phi H_{z0}^{+} + \eta\cos\theta H_{y0}^{+} \\ E_{x0}^{-} &= \eta\sin\theta\sin\phi H_{z0}^{-} - \eta\cos\theta H_{y0}^{-} \ , \end{aligned} \tag{2.209}$$

$$\begin{aligned} E_{y0}^{+} &= -\eta\cos\theta H_{x0}^{+} + \eta\sin\theta\cos\phi H_{z0}^{+} \\ E_{y0}^{-} &= \eta\cos\theta H_{x0}^{-} - \eta\sin\theta\cos\phi H_{z0}^{-} \ , \end{aligned} \tag{2.210}$$

$$\begin{aligned} E_{z0}^{+} &= -\eta\sin\theta\cos\phi H_{y0}^{+} + \eta\sin\theta\sin\phi H_{x0}^{+} \\ E_{z0}^{-} &= \eta\sin\theta\cos\phi H_{y0}^{-} - \eta\sin\theta\sin\phi H_{x0}^{-} \ , \end{aligned} \tag{2.211}$$

$$\begin{aligned} H_{x0}^{+} &= -\frac{1}{\eta}\cos\theta E_{y0}^{+} + \frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{+} \\ H_{x0}^{-} &= \frac{1}{\eta}\cos\theta E_{y0}^{-} - \frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{-} \ , \end{aligned} \tag{2.212}$$

$$H_{y0}^{+} = -\frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{+} + \frac{1}{\eta}\cos\theta E_{x0}^{+}$$
$$H_{y0}^{-} = \frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{-} - \frac{1}{\eta}\cos\theta E_{x0}^{-}\ , \tag{2.213}$$

and

$$H_{z0}^{+} = -\frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{+} + \frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{+}$$
$$H_{z0}^{-} = \frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{-} - \frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{-} \tag{2.214}$$

Combining identities (2.209) to (2.214) yields the following expressions

$$\sin^2\theta\cos^2\phi E_{x0}^{+} + \sin^2\theta\sin\phi\cos\phi E_{y0}^{+} + \sin\theta\cos\theta\cos\phi E_{z0}^{+} = 0$$
$$\sin^2\theta\cos^2\phi E_{x0}^{-} + \sin^2\theta\sin\phi\cos\phi E_{y0}^{-} + \sin\theta\cos\theta\cos\phi E_{z0}^{-} = 0 \tag{2.215}$$

and

$$\sin^2\theta\cos^2\phi H_{x0}^{+} + \sin^2\theta\sin\phi\cos\phi H_{y0}^{+} + \sin\theta\cos\theta\cos\phi H_{z0}^{+} = 0$$
$$\sin^2\theta\cos^2\phi H_{x0}^{-} + \sin^2\theta\sin\phi\cos\phi H_{y0}^{-} + \sin\theta\cos\theta\cos\phi H_{z0}^{-} = 0 \tag{2.216}$$

Applying the Fourier transform to the plane wave in (2.208) produces

$$\mathbf{G}(p,q,r,t) = \left\{ \begin{bmatrix} E_{x0}^{+} \\ E_{y0}^{+} \\ E_{z0}^{+} \\ H_{x0}^{+} \\ H_{y0}^{+} \\ H_{z0}^{+} \end{bmatrix} e^{-i2\pi\hat{k}vt} + \begin{bmatrix} E_{x0}^{-} \\ E_{y0}^{-} \\ E_{z0}^{-} \\ H_{x0}^{-} \\ H_{y0}^{-} \\ H_{z0}^{-} \end{bmatrix} e^{i2\pi\hat{k}vt} \right\} \delta\left(p - \hat{k}\sin\theta\cos\phi, q - \hat{k}\sin\theta\sin\phi, r - \hat{k}\cos\theta\right), \tag{2.217}$$

where

$$\mathbf{G}(p,q,r,t)=\begin{bmatrix} E_x(p,q,r,t) \\ E_y(p,q,r,t) \\ E_z(p,q,r,t) \\ H_x(p,q,r,t) \\ H_y(p,q,r,t) \\ H_z(p,q,r,t) \end{bmatrix}. \tag{2.218}$$

If our derivations have been correct thus far, then it must be the case that

$$\mathbf{G}(p,q,r,t)=\mathbf{H}(p,q,r,t-t_0)\mathbf{G}(p,q,r,t_0). \tag{2.219}$$

Applying the kernel matrix $\mathbf{H}(p,q,r,t-t_0)$ given in (2.173) to equation (2.217) yields

$$\begin{aligned} E_x(p,q,r,t)=\Big\{&\Big[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\Big] \\ &-\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}\left(q^2+r^2\right)\Big[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\Big] \\ &+\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}pq\Big[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\Big] \\ &+\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}pr\Big[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\Big] \\ &-ir\eta\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\Big[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\Big] \\ &+iq\eta\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\Big[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\Big]\Big\} \\ &\delta\left(p-\hat{k}\sin\theta\cos\phi,q-\hat{k}\sin\theta\sin\phi,r-\hat{k}\cos\theta\right), \end{aligned} \tag{2.220}$$

$$E_y(p,q,r,t)=\left\{\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}pq\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\right.$$

$$+E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}$$

$$-\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(p^2+r^2\right)\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]$$

$$+\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}qr\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right] \tag{2.221}$$

$$+ir\eta\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]$$

$$\left.-ip\eta\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\right\}$$

$$\delta\left(p-\hat{k}\sin\theta\cos\phi,q-\hat{k}\sin\theta\sin\phi,r-\hat{k}\cos\theta\right),$$

$$E_z(p,q,r,t)=\left\{\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}pr\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\right.$$

$$+\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}qr\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]$$

$$+E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}$$

$$-\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(p^2+q^2\right)\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \tag{2.222}$$

$$-iq\eta\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]$$

$$\left.+ip\eta\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\right\}$$

$$\delta\left(p-\hat{k}\sin\theta\cos\phi,q-\hat{k}\sin\theta\sin\phi,r-\hat{k}\cos\theta\right),$$

$$
\begin{aligned}
H_x(p,q,r,t) = \Bigg\{ & ir\frac{1}{\eta}\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& -iq\frac{1}{\eta}\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0} \\
& -\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}\left(q^2+r^2\right)\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}pq\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}pr\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\Bigg\} \\
& \delta\left(p-\hat{k}\sin\theta\cos\phi, q-\hat{k}\sin\theta\sin\phi, r-\hat{k}\cos\theta\right),
\end{aligned}
\tag{2.223}
$$

$$
\begin{aligned}
H_y(p,q,r,t) = \Bigg\{ & -ir\frac{1}{\eta}\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +ip\frac{1}{\eta}\frac{\sin\left(2\pi kv(t-t_0)\right)}{k}\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}pq\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0} \\
& -\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}\left(p^2+r^2\right)\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv(t-t_0)\right)\right)}{k^2}qr\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\Bigg\} \\
& \delta\left(p-\hat{k}\sin\theta\cos\phi, q-\hat{k}\sin\theta\sin\phi, r-\hat{k}\cos\theta\right),
\end{aligned}
\tag{2.224}
$$

and

$$
\begin{aligned}
H_z(p,q,r,t) = \Bigg\{ & iq\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& -ip\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}pr\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}qr\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right] \\
& +H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0} \\
& -\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(p^2+q^2\right)\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\Bigg\} \\
& \delta\left(p-\hat{k}\sin\theta\cos\phi,q-\hat{k}\sin\theta\sin\phi,r-\hat{k}\cos\theta\right).
\end{aligned}
\tag{2.225}
$$

Applying an inverse Fourier transform to equations (2.220) to (2.225) and using the identities (2.209) and (2.214) produces

$$
\begin{aligned}
&E_x(x,y,z,t)=\{\\
&E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\left(\hat{k}\sin\theta\sin\phi\right)^2+\left(\hat{k}\cos\theta\right)^2\right)\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\sin\theta\sin\phi\right)\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\cos\theta\right)\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&-i\eta\left(\hat{k}\cos\theta\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{+}+\frac{1}{\eta}\cos\theta E_{x0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\eta\left(\hat{k}\cos\theta\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{-}-\frac{1}{\eta}\cos\theta E_{x0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+i\eta\left(\hat{k}\sin\theta\sin\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{+}+\frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&+i\eta\left(\hat{k}\sin\theta\sin\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{-}-\frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)},
\end{aligned}
\tag{2.226}
$$

$$
\begin{aligned}
&E_y(x,y,z,t)=\{\\
&\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\sin\theta\sin\phi\right)\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\left(\hat{k}\sin\theta\cos\phi\right)^2+\left(\hat{k}\cos\theta\right)^2\right)\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\sin\phi\right)\left(\hat{k}\cos\theta\right)\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+i\left(\hat{k}\cos\theta\right)\eta\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\cos\theta E_{y0}^{+}+\frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&+i\left(\hat{k}\cos\theta\right)\eta\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\cos\theta E_{y0}^{-}-\frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\cos\phi\right)\eta\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{+}+\frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\cos\phi\right)\eta\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\sin\theta\sin\phi E_{x0}^{-}-\frac{1}{\eta}\sin\theta\cos\phi E_{y0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)},
\end{aligned}
\tag{2.227}
$$

$$
\begin{aligned}
&E_z(x,y,z,t)=\{\\
&\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\cos\theta\right)\left[E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\sin\phi\right)\left(\hat{k}\cos\theta\right)\left[E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi\hat{k}v(t-t_0)\right)\right)}{\hat{k}^2}\left(\left(\hat{k}\sin\theta\cos\phi\right)^2+\left(\hat{k}\sin\theta\sin\phi\right)^2\right)\left[E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&-i\eta\left(\hat{k}\sin\theta\sin\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\cos\theta E_{y0}^{+}+\frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\eta\left(\hat{k}\sin\theta\sin\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\cos\theta E_{y0}^{-}-\frac{1}{\eta}\sin\theta\sin\phi E_{z0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+i\eta\left(\hat{k}\sin\theta\cos\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(-\frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{+}+\frac{1}{\eta}\cos\theta E_{x0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&+i\eta\left(\hat{k}\sin\theta\cos\phi\right)\frac{\sin\left(2\pi\hat{k}v(t-t_0)\right)}{\hat{k}}\left(\frac{1}{\eta}\sin\theta\cos\phi E_{z0}^{-}-\frac{1}{\eta}\cos\theta E_{x0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)},
\end{aligned}
\tag{2.228}
$$

$$
\begin{aligned}
&H_x(x,y,z,t)=\{\\
&i\left(\hat{k}\cos\theta\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(-\eta\cos\theta H_{x0}^{+}+\eta\sin\theta\cos\phi H_{z0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&i\left(\hat{k}\cos\theta\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(\eta\cos\theta H_{x0}^{-}-\eta\sin\theta\cos\phi H_{z0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\sin\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(-\eta\sin\theta\cos\phi H_{y0}^{+}+\eta\sin\theta\sin\phi H_{x0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\sin\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(\eta\sin\theta\cos\phi H_{y0}^{-}-\eta\sin\theta\sin\phi H_{x0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\left(\hat{k}\sin\theta\sin\phi\right)^2+\left(\hat{k}\cos\theta\right)^2\right)\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\sin\theta\sin\phi\right)\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\cos\theta\right)\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&\}e^{i2\pi\hat{k}\left(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta\right)},
\end{aligned}
\tag{2.229}
$$

$$
\begin{aligned}
&H_y(x,y,z,t)=\{\\
&-i\left(\hat{k}\cos\theta\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(-\eta\sin\theta\sin\phi H_{z0}^{+}+\eta\cos\theta H_{y0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\cos\theta\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(\eta\sin\theta\sin\phi H_{z0}^{-}-\eta\cos\theta H_{y0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+i\left(\hat{k}\sin\theta\cos\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(-\eta\sin\theta\cos\phi H_{y0}^{+}+\eta\sin\theta\sin\phi H_{x0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&+i\left(\hat{k}\sin\theta\cos\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi\hat{k}v\left(t-t_0\right)\right)}{\hat{k}}\left(\eta\sin\theta\cos\phi H_{y0}^{-}-\eta\sin\theta\sin\phi H_{x0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\sin\theta\sin\phi\right)\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\left(\hat{k}\sin\theta\cos\phi\right)^2+\left(\hat{k}\cos\theta\right)^2\right)\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi\hat{k}v\left(t-t_0\right)\right)\right)}{\hat{k}^2}\left(\hat{k}\sin\theta\sin\phi\right)\left(\hat{k}\cos\theta\right)\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&\}e^{i2\pi\hat{k}\left(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta\right)},
\end{aligned}
\tag{2.230}
$$

and

$$
\begin{aligned}
&H_z(x,y,z,t) = \{\\
&i\left(\hat{k}\sin\theta\sin\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left(-\eta\sin\theta\sin\phi H_{z0}^{+}+\eta\cos\theta H_{y0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&i\left(\hat{k}\sin\theta\sin\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left(\eta\sin\theta\sin\phi H_{z0}^{-}-\eta\cos\theta H_{y0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\cos\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left(-\eta\cos\theta H_{x0}^{+}+\eta\sin\theta\cos\phi H_{z0}^{+}\right)e^{-i2\pi\hat{k}vt_0}\\
&-i\left(\hat{k}\sin\theta\cos\phi\right)\frac{1}{\eta}\frac{\sin\left(2\pi kv\left(t-t_0\right)\right)}{k}\left(\eta\cos\theta H_{x0}^{-}-\eta\sin\theta\cos\phi H_{z0}^{-}\right)e^{i2\pi\hat{k}vt_0}\\
&+\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(\hat{k}\sin\theta\cos\phi\right)\left(\hat{k}\cos\theta\right)\left[H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(\hat{k}\sin\theta\sin\phi\right)\hat{k}\cos\theta\left[H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\right]\\
&+H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\\
&-\frac{\left(1-\cos\left(2\pi kv\left(t-t_0\right)\right)\right)}{k^2}\left(\left(\hat{k}\sin\theta\cos\phi\right)^2+\left(\hat{k}\sin\theta\sin\phi\right)^2\right)\left[H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}+H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\right] \quad (2.231)\\
&\}e^{i2\pi\hat{k}\left(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta\right)}.
\end{aligned}
$$

Grouping the common terms in equations (2.227) to (2.231) yields

$$
\begin{aligned}
&E_x(x,y,z,t) = \{\\
&\left[e^{-i2\pi kv\left(t-t_0\right)}+\left(\sin\theta\cos\phi\right)^2\left(1-e^{-i2\pi kv\left(t-t_0\right)}\right)\right]E_{x0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{-i2\pi kv\left(t-t_0\right)}\right)\right]E_{y0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1-e^{-i2\pi kv\left(t-t_0\right)}\right)\right]E_{z0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[e^{i2\pi kv\left(t-t_0\right)}+\left(\sin\theta\cos\phi\right)^2\left(1-e^{i2\pi kv\left(t-t_0\right)}\right)\right]E_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{i2\pi kv\left(t-t_0\right)}\right)\right]E_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1-e^{i2\pi kv\left(t-t_0\right)}\right)\right]E_{z0}^{-}e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}\left(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta\right)},
\end{aligned}
\qquad (2.232)
$$

$$
\begin{aligned}
&E_y(x,y,z,t) = \{ \\
&\quad \left[\left(\sin^2\theta \sin\phi \cos\phi\right)\left(1 - e^{-2\pi\hat{k}v(t-t_0)}\right) \right] E_{x0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[e^{-i2\pi kv(t-t_0)} + \left(\sin\theta\sin\phi\right)^2 \left(1 - e^{-i2\pi kv(t-t_0)}\right) \right] E_{y0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1 - e^{-i2\pi kv(t-t_0)}\right) \right] E_{z0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin^2\theta \sin\phi \cos\phi\right)\left(1 - e^{2\pi\hat{k}v(t-t_0)}\right) \right] E_{x0}^{-} e^{i2\pi\hat{k}vt_0} \\
&+ \left[e^{i2\pi kv(t-t_0)} + \left(\sin\theta\sin\phi\right)^2 \left(1 - e^{i2\pi kv(t-t_0)}\right) \right] E_{y0}^{-} e^{i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1 - e^{i2\pi kv(t-t_0)}\right) \right] E_{z0}^{-} e^{i2\pi\hat{k}vt_0} \\
&\} e^{i2\pi\hat{k}(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta)},
\end{aligned}
\tag{2.233}
$$

$$
\begin{aligned}
&E_z(x,y,z,t) = \{ \\
&\quad \left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1 - e^{-2\pi\hat{k}v(t-t_0)}\right) \right] E_{x0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1 - e^{-i2\pi kv(t-t_0)}\right) \right] E_{y0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[e^{-i2\pi kv(t-t_0)} + \left(\cos\theta\right)^2 \left(1 - e^{-i2\pi kv(t-t_0)}\right) \right] E_{z0}^{+} e^{-i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1 - e^{2\pi\hat{k}v(t-t_0)}\right) \right] E_{x0}^{-} e^{i2\pi\hat{k}vt_0} \\
&+ \left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1 - e^{i2\pi kv(t-t_0)}\right) \right] E_{y0}^{-} e^{i2\pi\hat{k}vt_0} \\
&+ \left[e^{i2\pi kv(t-t_0)} + \left(\cos\theta\right)^2 \left(1 - e^{i2\pi kv(t-t_0)}\right) \right] E_{z0}^{-} e^{+i2\pi\hat{k}vt_0} \\
&\} e^{i2\pi\hat{k}(x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta)},
\end{aligned}
\tag{2.234}
$$

$$
\begin{aligned}
&H_x(x,y,z,t)=\{\\
&\quad\left[e^{-i2\pi kv(t-t_0)}+(\sin\theta\cos\phi)^2\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[e^{i2\pi kv(t-t_0)}+(\sin\theta\cos\phi)^2\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\cos\phi\right)\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)},
\end{aligned}
\tag{2.235}
$$

$$
\begin{aligned}
&H_y(x,y,z,t)=\{\\
&\quad\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{-2\pi\hat{k}v(t-t_0)}\right)\right]H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[e^{-i2\pi kv(t-t_0)}+(\sin\theta\sin\phi)^2\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin^2\theta\sin\phi\cos\phi\right)\left(1-e^{2\pi\hat{k}v(t-t_0)}\right)\right]H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[e^{i2\pi kv(t-t_0)}+(\sin\theta\sin\phi)^2\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[\left(\sin\theta\cos\theta\sin\phi\right)\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{z0}^{-}e^{i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)},
\end{aligned}
\tag{2.236}
$$

and

$$
\begin{aligned}
&H_z(x,y,z,t)=\{\\
&\left[(\sin\theta\cos\theta\cos\phi)\left(1-e^{-2\pi\hat{k}v(t-t_0)}\right)\right]H_{x0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[(\sin\theta\cos\theta\sin\phi)\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{y0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[e^{-i2\pi kv(t-t_0)}+(\cos\theta)^2\left(1-e^{-i2\pi kv(t-t_0)}\right)\right]H_{z0}^{+}e^{-i2\pi\hat{k}vt_0}\\
&+\left[(\sin\theta\cos\theta\cos\phi)\left(1-e^{2\pi\hat{k}v(t-t_0)}\right)\right]H_{x0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[(\sin\theta\cos\theta\sin\phi)\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{y0}^{-}e^{i2\pi\hat{k}vt_0}\\
&+\left[e^{i2\pi kv(t-t_0)}+(\cos\theta)^2\left(1-e^{i2\pi kv(t-t_0)}\right)\right]H_{z0}^{-}e^{+i2\pi\hat{k}vt_0}\\
&\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)}.
\end{aligned}
\tag{2.237}
$$

Finally, using identities (2.215) and (2.216) the preceding expressions can be written as

$$
\begin{bmatrix} E_x(x,y,z,t)\\ E_y(x,y,z,t)\\ E_z(x,y,z,t)\\ H_x(x,y,z,t)\\ H_y(x,y,z,t)\\ H_z(x,y,z,t) \end{bmatrix}
=\left\{\begin{bmatrix} E_{x0}^{+}\\ E_{y0}^{+}\\ E_{z0}^{+}\\ H_{x0}^{+}\\ H_{y0}^{+}\\ H_{z0}^{+} \end{bmatrix}e^{-i2\pi\hat{k}vt}+\begin{bmatrix} E_{x0}^{-}\\ E_{y0}^{-}\\ E_{z0}^{-}\\ H_{x0}^{-}\\ H_{y0}^{-}\\ H_{z0}^{-} \end{bmatrix}e^{i2\pi\hat{k}vt}\right\}e^{i2\pi\hat{k}(x\sin\theta\cos\phi+y\sin\theta\sin\phi+z\cos\theta)}. \tag{2.238}
$$

This result clearly shows that the matrix kernel applied to a three-dimensional wave correctly predicts its propagation.

2.4 Kernel Comparison

In this section a second look is taken at the kernels obtained in the preceding sections for a lossless homogeneous medium in one, two, and three dimensions. As will become apparent, there is a pattern to the matrices. The entries, although of different dimensionality, are surprisingly similar in structure.

By making the substitutions $F = \left(1-\cos\left(2\pi pvt\right)\right)/p^2$ and $G = \sin\left(2\pi pvt\right)/p$, the one-dimensional kernel matrix in inverse-space $\mathbf{H}\left(p,t\right)$ can be written as

$$\mathbf{H}\left(p,t\right) = \begin{bmatrix} 1-p^2F & -i\eta pG \\ -ipG/\eta & 1-p^2F \end{bmatrix}. \tag{2.239}$$

The two- and three-dimensional kernel matrices are repeated below for the sake of comparison

$$\mathbf{H}\left(p,q,t\right) = \begin{bmatrix} 1-\left(p^2+q^2\right)F & -i\eta qG & i\eta pG \\ -iqG/\eta & 1-q^2F & pqF \\ ipG/\eta & pqF & 1-p^2F \end{bmatrix} \tag{2.240}$$

and

$$\mathbf{H}(p,q,r,t)=$$
$$\begin{bmatrix} 1-(q^2+r^2)F & pqF & prF & 0 & -ir\eta G & iq\eta G \\ pqF & 1-(p^2+r^2)F & qrF & ir\eta G & 0 & -ip\eta G \\ prF & qrF & 1-(p^2+q^2)F & -iq\eta G & ip\eta G & 0 \\ 0 & irG/\eta & -iqG/\eta & 1-(q^2+r^2)F & pqF & prF \\ -irG/\eta & 0 & ipG/\eta & pqF & 1-(p^2+r^2)F & qrF \\ iqG/\eta & -ipG/\eta & 0 & prF & qrF & 1-(p^2+q^2)F \end{bmatrix}, \tag{2.241}$$

where $F=(1-\cos(2\pi k\upsilon t))/k^2$, $G=\sin(2\pi k\upsilon t)/k$, $k=\sqrt{p^2+q^2}$ for $\mathbf{H}(p,q,t)$, and $k=\sqrt{p^2+q^2+r^2}$ for $\mathbf{H}(p,q,r,t)$.

When shown side by side, the similarities among the kernel matrices become apparent. As the dimensionality increases or decreases the elements maintain their basic form.

Chapter 3

Discrete Solutions to Maxwell's Equations

In the previous chapter, analytical solutions were obtained for both the homogeneous medium one-dimensional form of Maxwell's equations and the uniform one-dimensional transmission line. In this chapter, these one-dimensional solutions are converted from continuous to discrete form. Examples of propagating waves and transmission lines are simulated using MATLAB scripts that implement these discrete solutions. Several snapshots of the simulations are included. Towards the end of the chapter, a method that can be used to obtain similar discrete forms for the two- and three-dimensional solutions is discussed.

3.1 Discrete Algorithm for the One-Dimensional Case

In section 2.2.1 the analytic solution to the one-dimensional form of Maxwell's equation for a lossless homogeneous medium was derived. In this derivation, an x-directed, y-polarized one dimensional plane wave with components $E_y(x,t)$ and $H_z(x,t)$ was used. The solution is repeated below for convenience:

$$
\begin{aligned}
E_y(x,t) = & \frac{1}{2}\left[E_y\left(x+v(t-t_0),t_0\right)+E_y\left(x-v(t-t_0),t_0\right)\right] \\
- & \frac{\eta}{2}\left[H_z\left(x+v(t-t_0),t_0\right)-H_z\left(x-v(t-t_0),t_0\right)\right] \\
- & \int_{t_0}^{t}\frac{1}{2\varepsilon}\left[J_0\left(x+v(t-\tau),\tau\right)+J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\
+ & \int_{t_0}^{t}\frac{\eta}{2\mu}\left[M_0\left(x+v(t-\tau),\tau\right)-M_0\left(x-v(t-\tau),\tau\right)\right]d\tau
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
H_z(x,t) = & -\frac{1}{2\eta}\left[E_y\left(x+v(t-t_0),t_0\right)-E_y\left(x-v(t-t_0),t_0\right)\right] \\
+ & \frac{1}{2}\left[H_z\left(x+v(t-t_0),t_0\right)+H_z\left(x-v(t-t_0),t_0\right)\right] \\
+ & \int_{t_0}^{t}\frac{1}{2\eta\varepsilon}\left[J_0\left(x+v(t-\tau),\tau\right)-J_0\left(x-v(t-\tau),\tau\right)\right]d\tau \\
- & \int_{t_0}^{t}\frac{1}{2\mu}\left[M_0\left(x+v(t-\tau),\tau\right)+M_0\left(x-v(t-\tau),\tau\right)\right]d\tau
\end{aligned}
\tag{3.2}
$$

To obtain discrete versions of equations (3.1) and (3.2) time and space are partitioned in steps of Δx and Δt, respectively. By setting $x = k\Delta x$, $t_0 = n\Delta x$, and $t = (n+1)\Delta x$, every update of the equations now corresponds to Δt time units. By setting $v\Delta t = \Delta x$ things can be simplified even further, so that waves propagating at speed v travel Δx space units in Δt time units. Substituting into equations (3.1) and (3.2) yields

$$
\begin{aligned}
E_y\left(k\Delta x,(n+1)\Delta t\right) = & \frac{1}{2}\left[E_y\left((k+1)\Delta x,n\Delta t\right)+E_y\left((k-1)\Delta x,n\Delta t\right)\right] \\
- & \frac{\eta}{2}\left[H_z\left((k+1)\Delta x,n\Delta t\right)-H_z\left((k-1)\Delta x,n\Delta t\right)\right] \\
- & \frac{\Delta t}{2\varepsilon}\left[J_0\left((k+1)\Delta x,n\Delta t\right)+J_0\left((k-1)\Delta x,n\Delta t\right)\right] \\
+ & \frac{\eta\Delta t}{2\mu}\left[M_0\left((k+1)\Delta x,n\Delta t\right)-M_0\left((k-1)\Delta x,n\Delta t\right)\right]
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
H_z\left(k\Delta x,(n+1)\Delta t\right) = & -\frac{1}{2\eta}\left[E_y\left((k+1)\Delta x, n\Delta t\right) - E_y\left((k-1)\Delta x, n\Delta t\right)\right] \\
& + \frac{1}{2}\left[H_z\left((k+1)\Delta x, n\Delta t\right) + H_z\left((k-1)\Delta x, n\Delta t\right)\right] \\
& + \frac{\Delta t}{2\eta\varepsilon}\left[J_0\left((k+1)\Delta x, n\Delta t\right) - J_0\left((k-1)\Delta x, n\Delta t\right)\right] \\
& - \frac{\Delta t}{2\mu}\left[M_0\left((k+1)\Delta x, n\Delta t\right) + M_0\left((k-1)\Delta x, n\Delta t\right)\right].
\end{aligned}
\tag{3.4}
$$

Equations (3.3) and (3.4) do not include any boundary conditions. In order to have a useful algorithm the medium must be self-contained. If the region has a total of n locations, three regions are distinguished: the left edge $k=1$, the middle zone $k=2,\ldots,n-1$, and the right edge $k=n$.

Using a shorthand notation, the update for the middle zone (locations $k=2,\ldots,n-1$) is

$$
\begin{aligned}
E_y(k,n+1) = & \frac{1}{2}\left[E_y(k+1,n) + E_y(k-1,n)\right] - \frac{\eta}{2}\left[H_z(k+1,n) - H_z(k-1,n)\right] \\
& - \frac{\Delta t}{2\varepsilon}\left[J_0(k+1,n) + J_0(k-1,n)\right] + \frac{\eta\Delta t}{2\mu}\left[M_0(k+1,n) - M_0(k-1,n)\right]
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
H_z(k,n+1) = & -\frac{1}{2\eta}\left[E_y(k+1,n) - E_y(k-1,n)\right] + \frac{1}{2}\left[H_z(k+1,n) + H_z(k-1,n)\right] \\
& + \frac{\Delta t}{2\eta\varepsilon}\left[J_0(k+1,n) - J_0(k-1,n)\right] - \frac{\Delta t}{2\mu}\left[M_0(k+1,n) + M_0(k-1,n)\right].
\end{aligned}
\tag{3.6}
$$

For the left edge, location $(k=1)$, the update is given by

$$
E(1,n+1) = \frac{1}{2}\left[E(2,n) - \eta H(2,n)\right] + \frac{\Delta t}{2}\left[-\frac{1}{\varepsilon}J_0(2,n) + \frac{\eta}{\mu}M_0(2,n)\right]
\tag{3.7}
$$

and

$$H(1,n+1)=\frac{1}{2}\left[-\frac{1}{\eta}E(2,n)+H(2,n)\right]+\frac{\Delta t}{2}\left[\frac{1}{\eta\varepsilon}J_0(2,n)-\frac{1}{\mu}M_0(2,n)\right]. \quad (3.8)$$

For the right edge, location $(k=n)$, the update is

$$E(n,n+1)=\frac{1}{2}\left[E(n-1,n)-\eta H(n-1,n)\right]-\frac{\Delta t}{2}\left[\frac{1}{\varepsilon}J_0(n-1,n)+\frac{\eta}{\mu}M_0(n-1,n)\right] \quad (3.9)$$

and

$$H(n,n+1)=\frac{1}{2}\left[\frac{1}{\eta}E(n-1,n)+H(n-1,n)\right]-\frac{\Delta t}{2}\left[\frac{1}{\eta\varepsilon}J_0(n-1,n)+\frac{1}{\mu}M_0(n-1,n)\right]. \quad (3.10)$$

The preceding updates only apply when the medium is homogeneous. If it is desired to simulate scattering by cascading two regions, each uniform but having different properties, then the update at the boundary between the regions must include electric and magnetic field components from both sides.

3.1.1 Traveling Pulse

The case of pulse propagating in a medium composed of two adjoining regions of different permittivities is considered in this section. Since the permittivities are different, the pulse splits into a transmitted and a reflected pulse upon reaching the junction. The width and the speed of propagation of these new pulses is a function of the permittivities of the regions they travel on and thus are different.

For this example, a Gaussian pulse is used as the source for the electric field. The permittivities for the regions are $\varepsilon_1 = 11\varepsilon_0$ and $\varepsilon_2 = 1.7\varepsilon_0$. Using the MATLAB program `wave1d2.m` the pulse can be observed as it propagates and then morphs into the transmitted and reflected pulses as it crosses the junction.

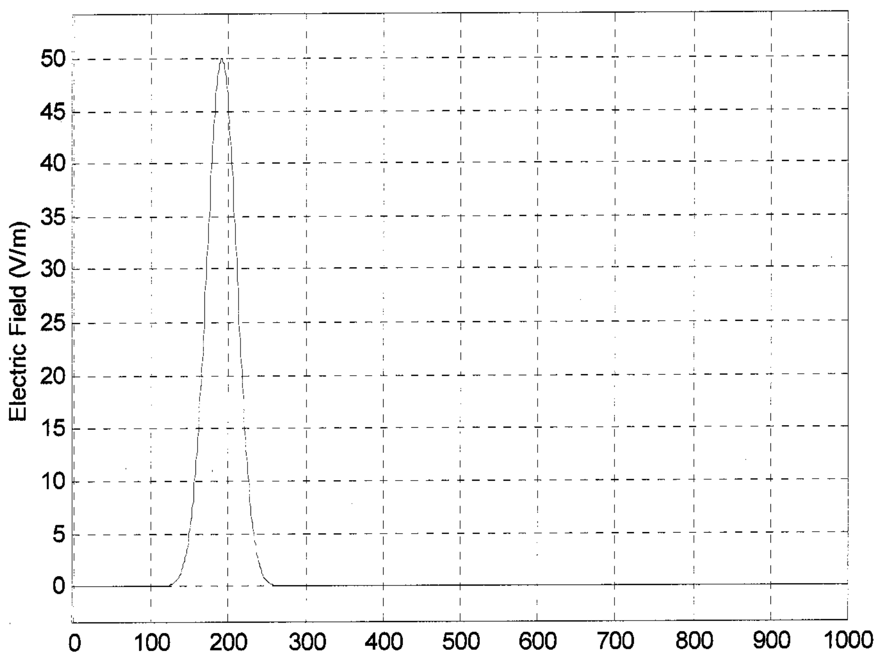

Figure 3.1: Electric field pulse as it travels through the first region

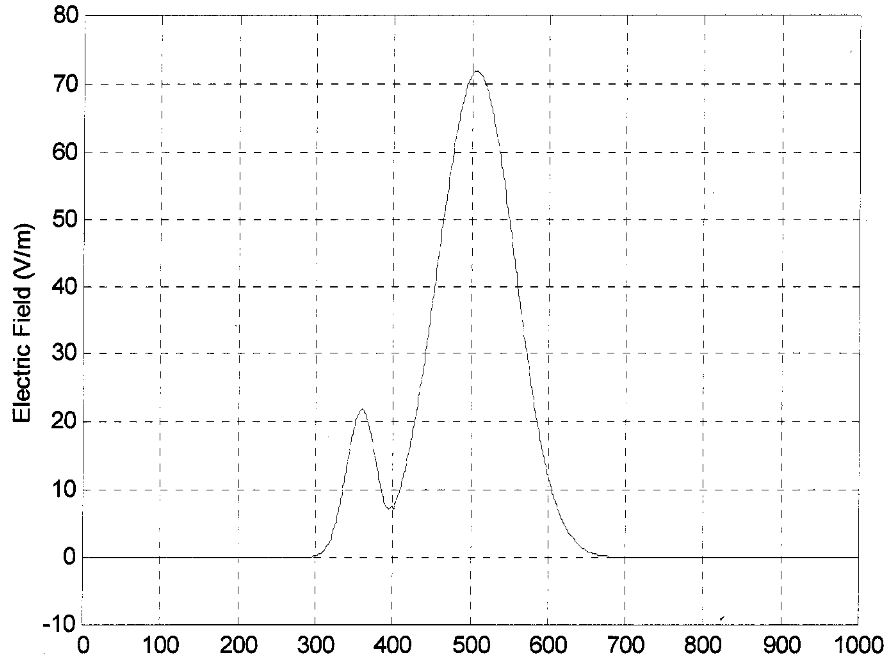

Figure 3.2: Electric field pulse as it splits at the junction

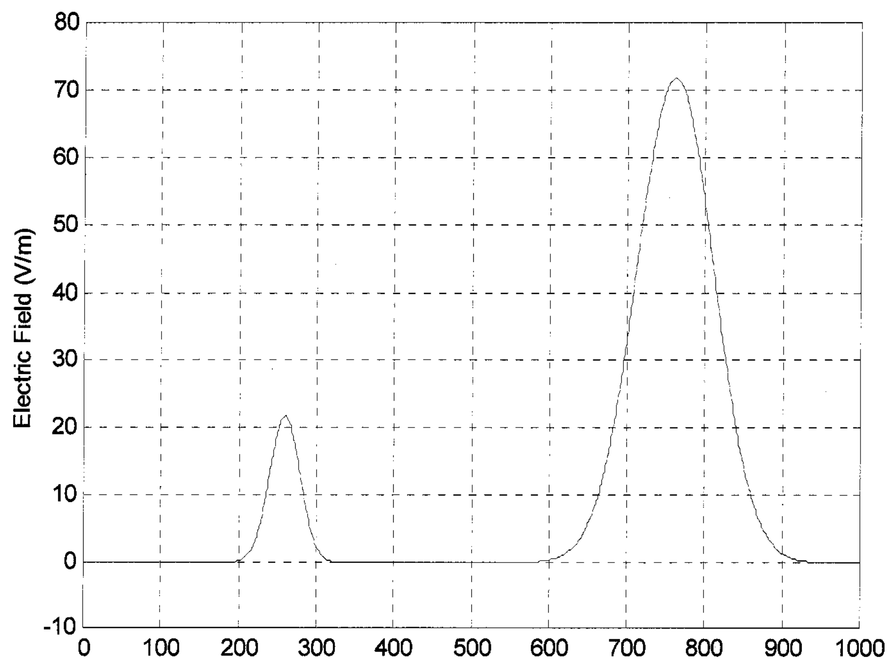

Figure 3.3: Transmitted and reflected electric field waves

For this particular example, the transmitted pulse is expected to be wider and to travel faster than the reflected pulse, because of the choice of permittivities. The plots above are in accordance with the expected results.

3.2 Discrete Algorithm for the One-Dimensional Transmission Line

In section 2.2.4 the solution to the lossless transmission line equations was obtained by using essentially the same approach used to solve the one-dimensional form of Maxwell's equations. The solutions obtained in that section are repeated below for convenience:

$$\begin{aligned} V(x,t) = & \frac{1}{2}\left[V\left(x+\upsilon(t-t_0),t_0\right)+V\left(x-\upsilon(t-t_0),t_0\right)\right] \\ & - \frac{Z_0}{2}\left[I\left(x+\upsilon(t-t_0),t_0\right)-I\left(x-\upsilon(t-t_0),t_0\right)\right] \\ & - \int_0^t \frac{1}{2C}\left[J_0\left(x+\upsilon(t-\tau),\tau\right)+J_0\left(x-\upsilon(t-\tau),\tau\right)\right]d\tau \\ & + \int_0^t \frac{Z_0}{2L}\left[E_0\left(x+\upsilon(t-\tau),\tau\right)-E_0\left(x-\upsilon(t-\tau),\tau\right)\right]d\tau \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} I(x,t) = & -\frac{1}{2Z_0}\left[V\left(x+\upsilon(t-t_0),t_0\right)-V\left(x-\upsilon(t-t_0),t_0\right)\right] \\ & + \frac{1}{2}\left[I\left(x+\upsilon(t-t_0),t_0\right)+I\left(x-\upsilon(t-t_0),t_0\right)\right] \\ & + \int_0^t \frac{1}{2Z_0C}\left[J_0\left(x+\upsilon(t-\tau),\tau\right)-J_0\left(x-\upsilon(t-\tau),\tau\right)\right]d\tau \\ & - \int_0^t \frac{1}{2L}\left[E_0\left(x+\upsilon(t-\tau),\tau\right)+E_0\left(x-\upsilon(t-\tau),\tau\right)\right]d\tau. \end{aligned} \tag{3.12}$$

To obtain discrete versions of equations (3.11) and (3.12) the same approach is used as in section 3.1. Time and space are partitioned in steps of Δx and Δt, respectively, and the assignments $x = k\Delta x$, $t_0 = n\Delta x$, $t = (n+1)\Delta x$, and $v\Delta t = \Delta x$ are made. The resulting discrete equations are

$$\begin{aligned} V(k\Delta x,(n+1)\Delta t) = &\ \frac{1}{2}\left[V((k+1)\Delta x, n\Delta t) + V((k-1)\Delta x, n\Delta t)\right] \\ - &\ \frac{Z_0}{2}\left[I((k+1)\Delta x, n\Delta t) - I((k-1)\Delta x, n\Delta t)\right] \\ - &\ \frac{\Delta t}{2C}\left[J_0((k+1)\Delta x, n\Delta t) + J_0((k-1)\Delta x, n\Delta t)\right] \\ + &\ \frac{Z_0\Delta t}{2L}\left[E_0((k+1)\Delta x, n\Delta t) - E_0((k-1)\Delta x, n\Delta t)\right] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I(k\Delta x,(n+1)\Delta t) = &\ -\frac{1}{2Z_0}\left[V((k+1)\Delta x, n\Delta t) - V((k-1)\Delta x, n\Delta t)\right] \\ + &\ \frac{1}{2}\left[I((k+1)\Delta x, n\Delta t) + I((k-1)\Delta x, n\Delta t)\right] \\ + &\ \frac{\Delta t}{2Z_0C}\left[J_0((k+1)\Delta x, n\Delta t) - J_0((k-1)\Delta x, n\Delta t)\right] \\ - &\ \frac{\Delta t}{2L}\left[E_0((k+1)\Delta x, n\Delta t) + E_0((k-1)\Delta x, n\Delta t)\right]. \end{aligned} \tag{3.14}$$

Using the shorthand notation, the equations can be written as

$$\begin{aligned} V(k,n+1) = &\ \frac{1}{2}\left[V(k+1,n) + V(k-1,n)\right] - \frac{Z_0}{2}\left[I(k+1,n) - I(k-1,n)\right] \\ &- \frac{\Delta t}{2C}\left[J_0(k+1,n) + J_0(k-1,n)\right] + \frac{Z_0\Delta t}{2L}\left[E_0(k+1,n) - E_0(k-1,n)\right] \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} I(k,n+1) = &\ -\frac{1}{2Z_0}\left[V(k+1,n) - V(k-1,n)\right] + \frac{1}{2}\left[I(k+1,n) + I(k-1,n)\right] \\ &+ \frac{\Delta t}{2Z_0C}\left[J_0(k+1,n) - J_0(k-1,n)\right] - \frac{\Delta t}{2L}\left[E_0(k+1,n) + E_0(k-1,n)\right]. \end{aligned} \tag{3.16}$$

Not all the locations receive the same update, the endpoints of a region need to be updated differently to properly account for the transition. In the sections that follow we show examples with a generator and a load, and with cascaded transmission lines of different characteristic impedances.

3.2.1 Transmission Line with a Load and a Generator

In this case, the transmission line is connected to a resistive voltage source on the left side (at location $k=1$) and terminated with a load on the right side (at location $k=m$). While locations $k=2,...,m-1$ are updated using equations (3.15) and (3.16), the update equations for the boundary points are modified to account for these elements. The update at the generator has to include the voltage/current from the source and the incoming voltage/current from its immediate neighbor. The voltage/current from the source is divided according to the source resistance and the impedance of the transmission line. The incoming voltage/current is multiplied by the transmission coefficient. The resulting update equations at the generator $(k=1)$ are given by

$$V(1,n+1)=\left[\frac{1}{2}V(2,n)-\frac{Z_0}{2}I(2,n)\right](1+\rho_G)+Z_0\left(\frac{V_G(n+1)}{Z_0+Z_G}\right) \tag{3.17}$$

and

$$I(1,n+1)=\left[-\frac{1}{2Z_0}V(2,n)+\frac{1}{2}I(2,n)\right](1-\rho_G)+\left(\frac{V_G(n+1)}{Z_0+Z_G}\right), \tag{3.18}$$

where $\rho_G=\dfrac{Z_0-Z_G}{Z_0+Z_G}$.

The update equation at the load only needs to include the incoming voltage/current from its immediate neighbor. This incoming voltage/current is multiplied by the transmission coefficient. The resulting update equations at the load $(k = m)$ are given by

$$V(m,n+1) = \left[\frac{1}{2}V(m-1,n) + \frac{Z_0}{2}I(m-1,n)\right](1+\rho_L) \tag{3.19}$$

and

$$I(m,n+1) = \left[\frac{1}{2Z_0}V(m-1,n) + \frac{1}{2}I(m-1,n)\right](1-\rho_L), \tag{3.20}$$

where $\rho_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$.

The first example is a transmission line with parameters $L = 0.5uH/m$, $C = 200pF/m$, a load resistance $R_L = 75$, and a resistive voltage source with a resistance $R_G = 25$. The length of the transmission line is 1000 meters, and it is partitioned into $n = 100$ locations ($\Delta x = 10m$). The voltage applied is DC of magnitude $V_G = 10V$. The MATLAB script used for the simulation is `tline_demo1.m`. The voltage and current at locations 50 (the middle of the transmission line) and 100 (the load) are monitored and the following plots obtained.

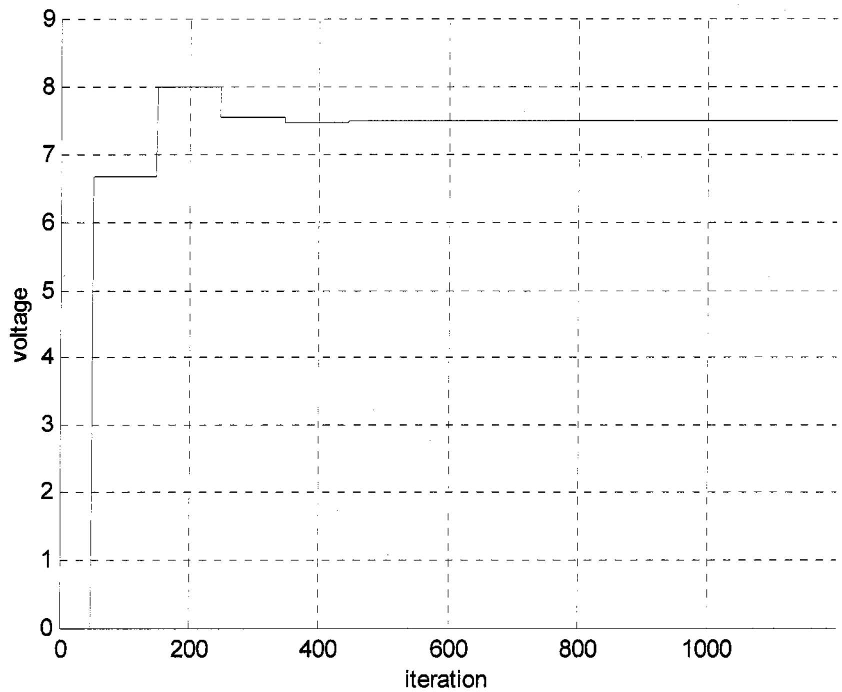

Figure 3.4: Voltage at location 50 (the middle of the transmission line)

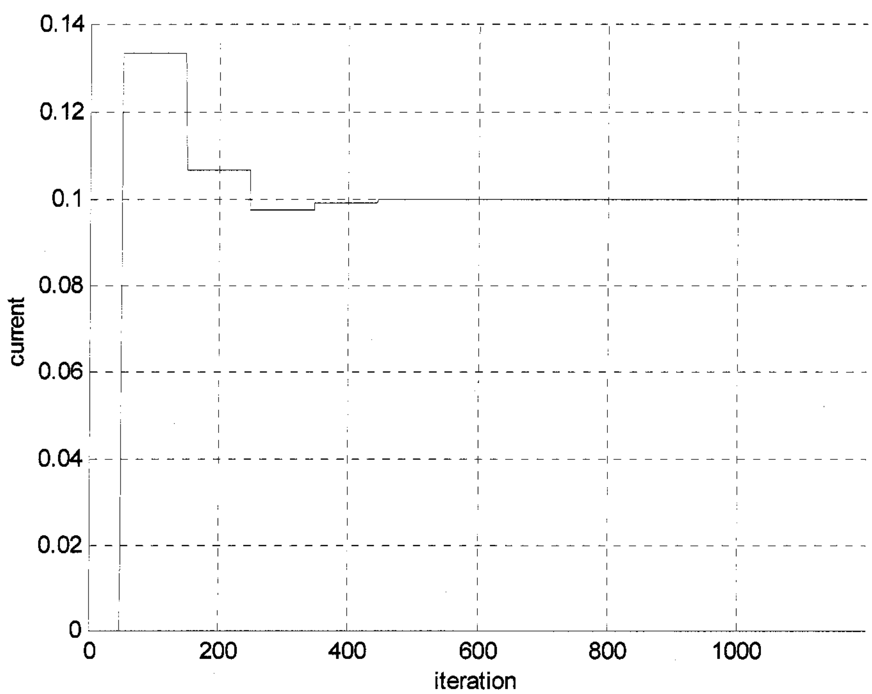

Figure 3.5: Current at location 50 (the middle of the transmission line)

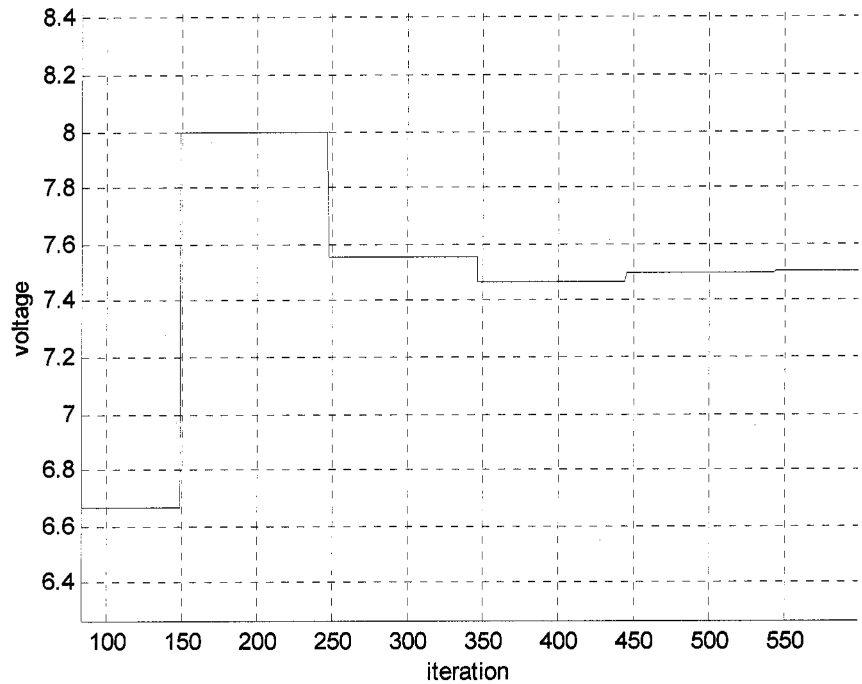

Figure 3.6: Closeup view of the voltage at location 50 (the middle of the transmission line)

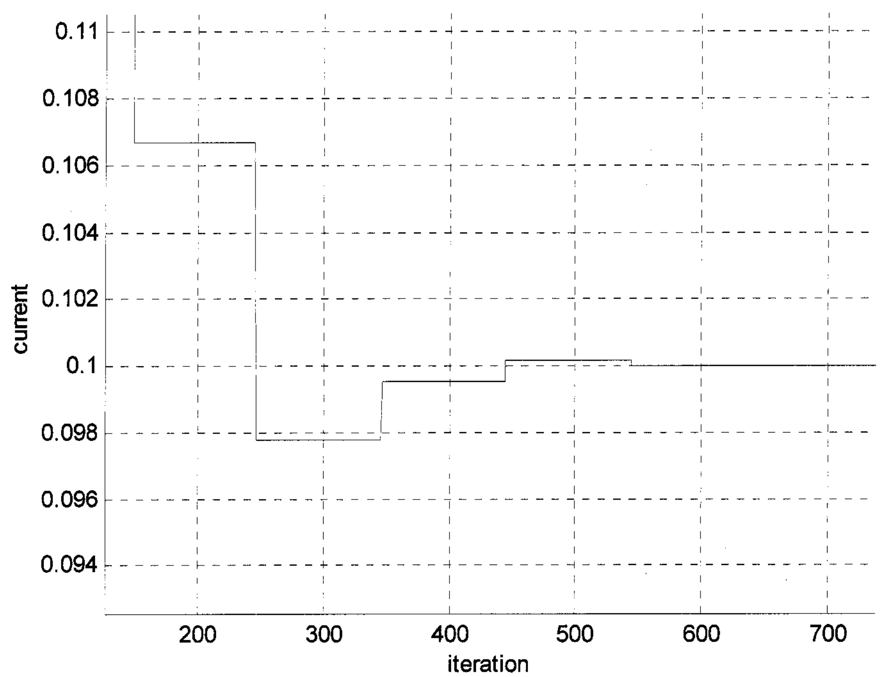

Figure 3.7: Closeup view of the current at location 50 (the middle of the transmission line)

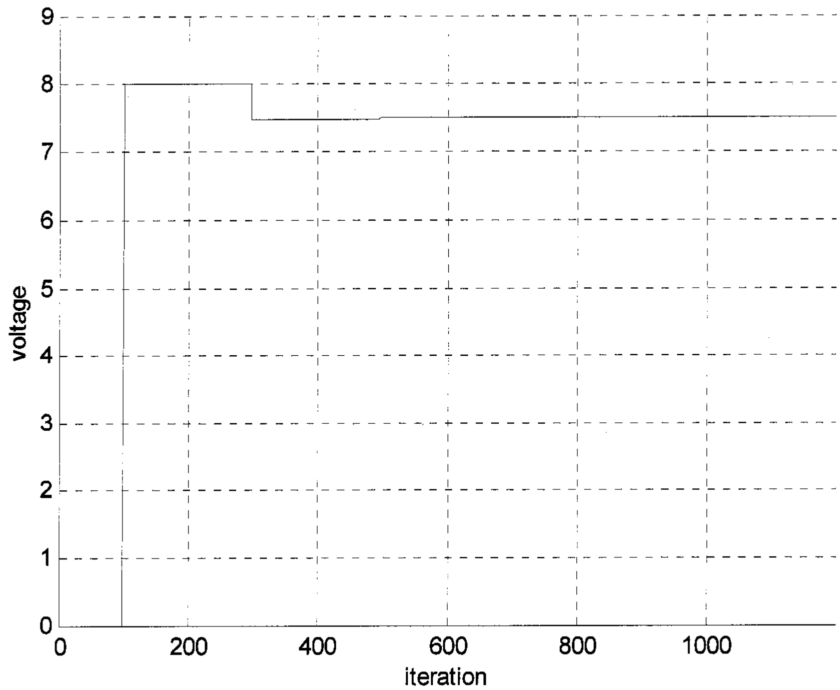

Figure 3.8: Voltage at location 100 (the load)

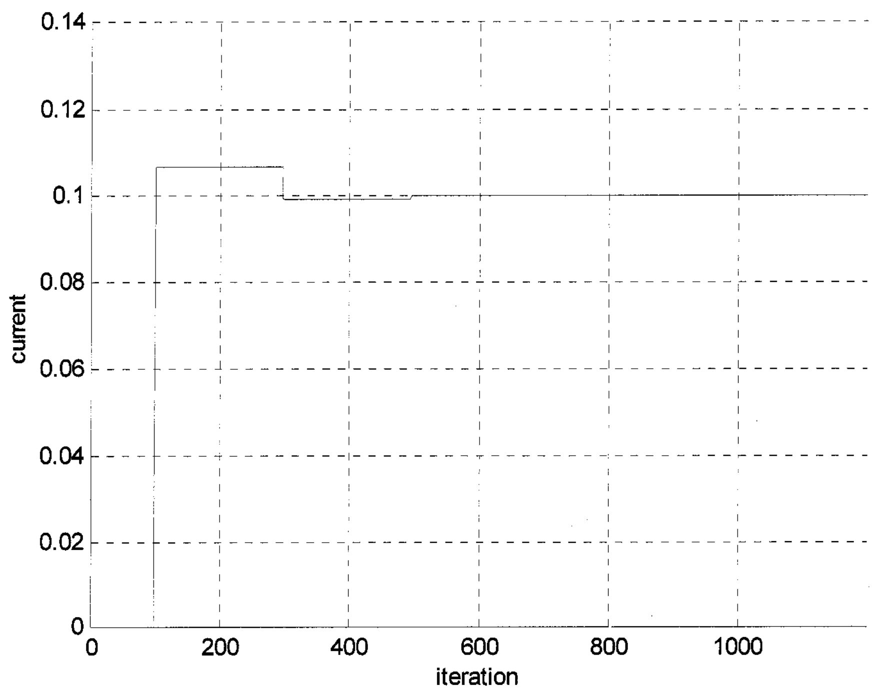

Figure 3.9: Current at location 100 (the load)

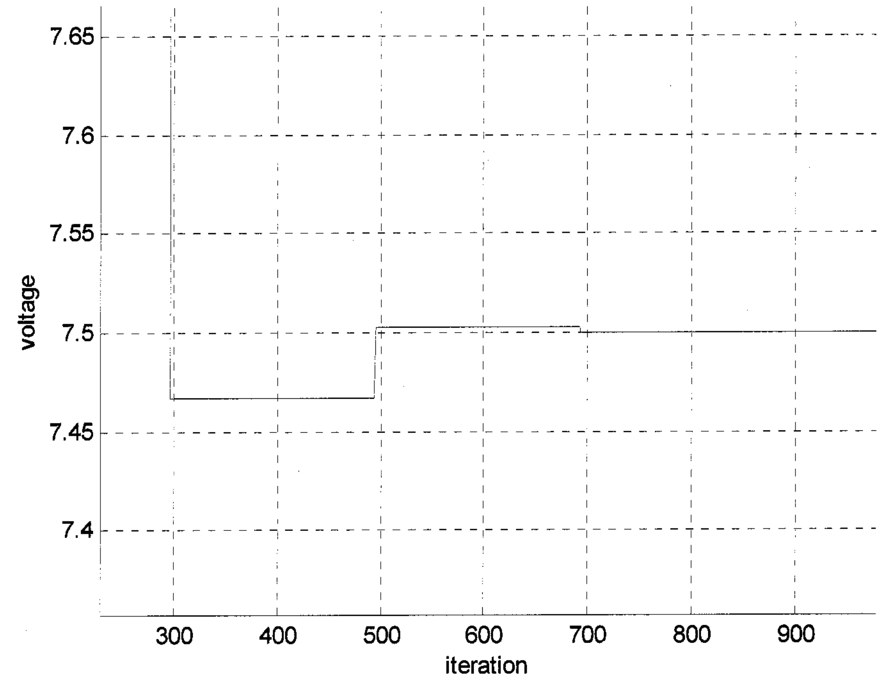

Figure 3.10: Closeup view of the voltage at location 100 (the load)

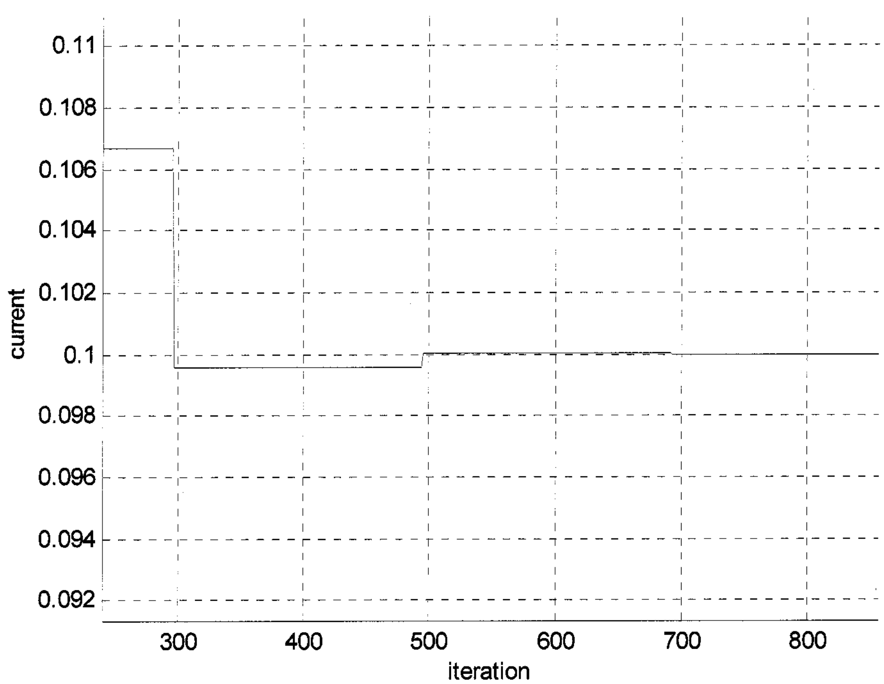

Figure 3.11: Closeup view of the current at location 100 (the load)

The results from the plots are exactly what is expected, as can easily be verified by drawing a bounce diagram.

The next example uses the same transmission line, a short-circuited load resistance $R_L = 0$, and a resistive voltage source with a resistance $R_G = 25$. The applied voltage is a sinusoid $V_G = 10\cos(2\pi f_0 t)V$, where $f_0 = 0.1Mhz$. For this case the voltage at location 100 (the load) should be zero and standing waves should be observed once a steady state is reached, that is, the amplitude should move up and down, but no movement of the wave either to the left or right should be observable.

For this example the Matlab file `tline_demo2.m` is used with a sampling frequency $f_s = 4Mhz$ (40 samples per period) as input. The following plots are obtained (the voltages and currents corresponding to different iterations are plotted using different lines to ease visualization).

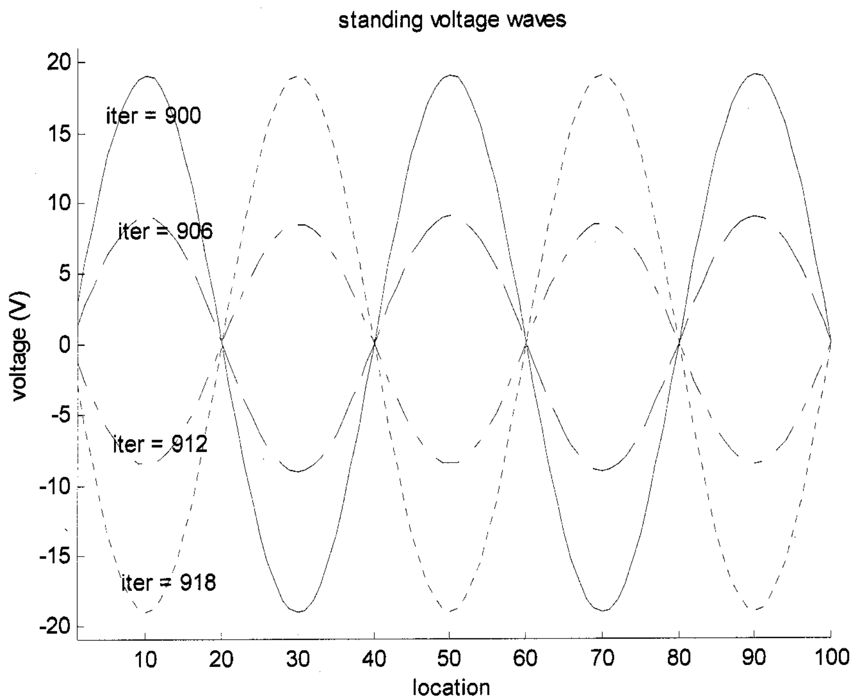

Figure 3.12: Voltage in steady state at different iterations

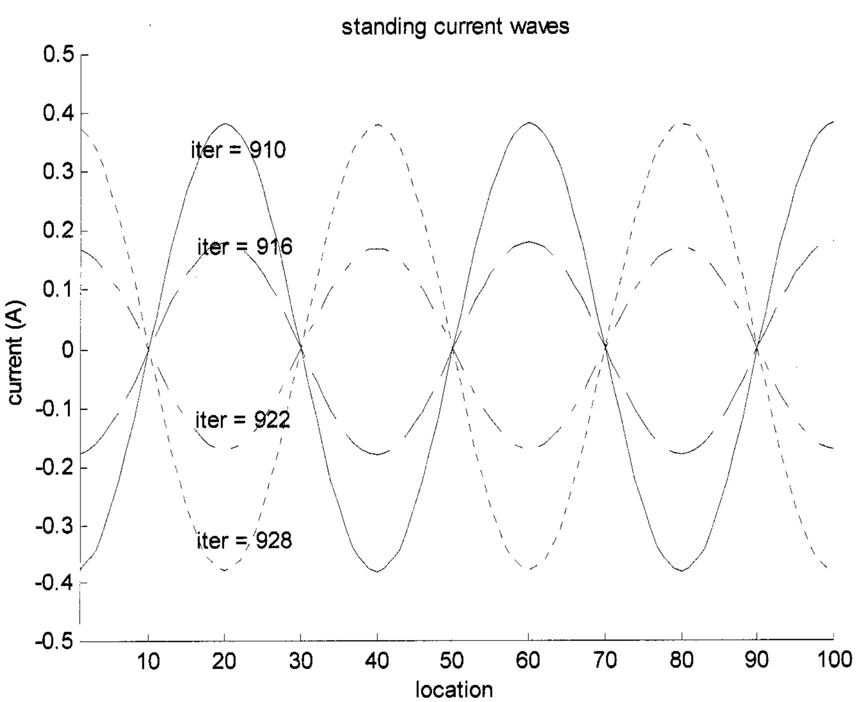

Figure 3.13: Current in steady state at different iterations

After 900 iterations steady state has, for practical purposes, been reached. As can be seen from the plots, the amplitude swings up and down, but there is no observable left or right movement of the waves.

3.2.2 Cascaded Transmission Lines

When transmission lines of different characteristic impedances are cascaded it is expected that reflections will occur whenever signals arrive at the junction. Part of the signal is transmitted and part is reflected. In the case of a pulse, one of the pulses will be narrower and will travel at a lower speed that the other pulse.

To illustrate the method for this case, two transmission lines are adjoined. The first one (on the left hand side) is attached to a resistive voltage source on its left edge. The second one (on the right hand side) is terminated with a resistive load on its right edge. The first transmission line has parameters $L_1 = 0.5uH/m$, $C_1 = 200pF/m$, which correspond to an impedance of $Z_1 = 50\Omega$, and a propagation speed $v_1 = 10^8 m/s$. The second transmission line has parameters $L_2 = 0.5uH/m$, $C_2 = 50pF/m$, which correspond to an impedance of $Z_1 = 100\Omega$, and a propagation speed $v_1 = 2*10^8 m/s$. The load resistance is $R_L = 50$, and the voltage source has a resistance $R_G = 50$. The first transmission line has a length of 1000 meters, and it is partitioned into $n = 400$ locations ($\Delta x_1 = 2.5m$). The second transmission line has a length of 3000 meters, and it is partitioned into $n = 800$ locations ($\Delta x_2 = 3.75m$). The applied voltage is a pulse of amplitude $V_G = 5V$. Using the Matlab code `tline2.m`, the current and voltage pulses can be observed as they evolve in time, in the following plots:

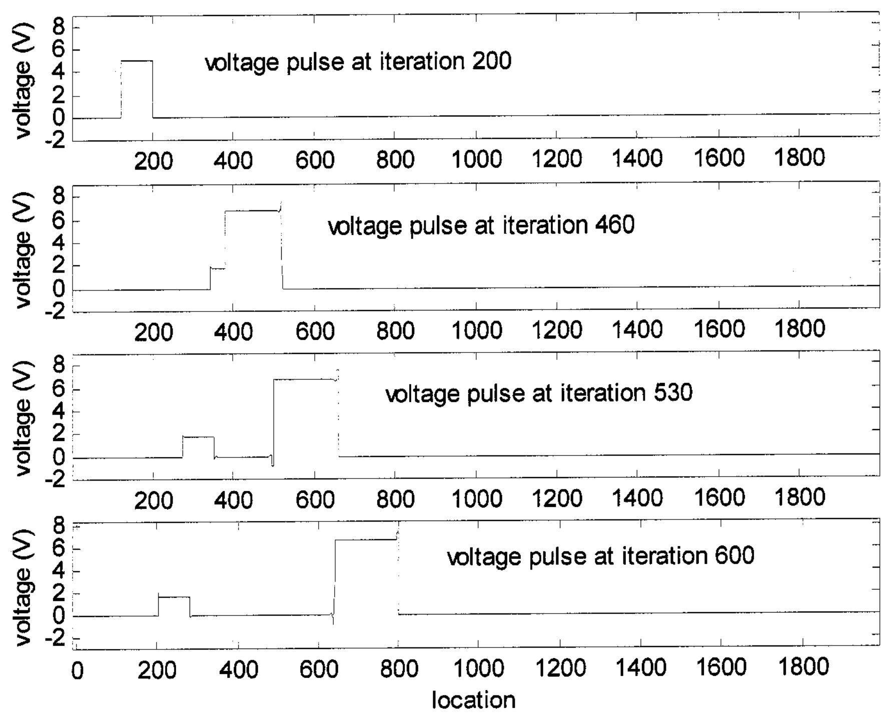

Figure 3.14: Voltage pulse as it travels through the cascaded transmission lines

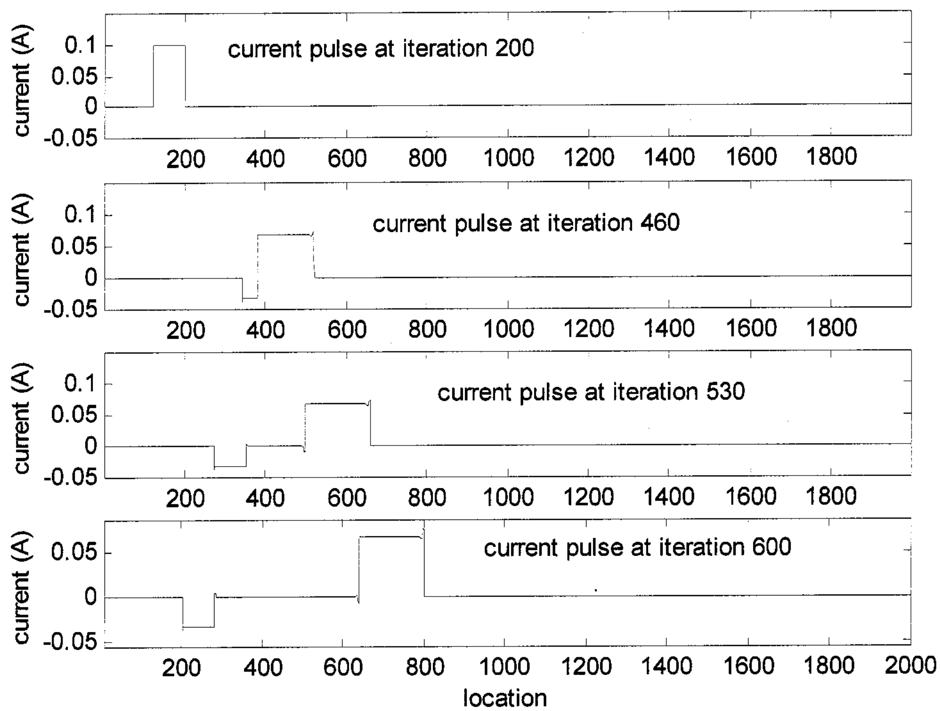

Figure 3.15: Current pulse as it travels through the cascaded transmission lines

The results correlate quite well with the expectations. Since the impedance of the second transmission line is larger than that of the first one, pulses on transmission line two should be wider and travel faster than pulses on transmission line one. This is exactly what the plots show.

3.3 Discrete Solutions for Higher Dimensions

The sampling theorem can be used to transform the continuous analytical solutions obtained in chapter two into a discrete form. This applies to the one-, two-, and three-dimensional cases.

Consider a signal $g(x)$ whose Fourier transform

$$G(k)=\int_{-\infty}^{\infty} g(x)e^{-i2\pi kx}dx \tag{3.21}$$

vanishes on the range $|k|\geq 1/(2\Delta x)$. The sampling theorem guarantees that this signal can be exactly reconstructed from its individual samples as long as these are separated by Δx or less. Formally this is stated below

$$g(x)=\sum_{l=-N}^{N} g(x_l)\operatorname{sinc}\left(\frac{x-x_l}{\Delta x}\right)+e_N(x). \tag{3.22}$$

The term $e_N(x)$ is the error incurred as a result of truncating the signal to the interval $(-N,N)$. The error becomes zero as $N\to\infty$.

It is relevant to point out that $G(k)$ need not exactly vanish for $|k|\geq 1/2\Delta x$. If the signal is sufficiently small on this interval the extra error incurred may be acceptable for the application at hand. In that case the sampling theorem can be stated as

$$g(x)=\sum_{l=-N}^{N} g(x_l)\operatorname{sinc}\left(\frac{x-x_l}{\Delta x}\right)+e_N(x,\Delta x), \tag{3.23}$$

where the error term is now a function of N, x, and Δx.

The same criterion applies to higher dimensions. The formulas for the one- and two-dimensional cases are

$$g(x,y)=\sum_{l=-N}^{N}\sum_{m=-N}^{N} g(x_l,y_m)\,\mathrm{sinc}\left(\frac{x-x_l}{\Delta x}\right)\mathrm{sinc}\left(\frac{y-y_m}{\Delta y}\right)+e_N(x,y,\Delta x,\Delta y) \tag{3.24}$$

and

$$\begin{aligned}g(x,y,z)=&\sum_{l=-N}^{N}\sum_{m=-N}^{N}\sum_{n=-N}^{N} g(x_l,y_m,z_n)\,\mathrm{sinc}\left(\frac{x-x_l}{\Delta x}\right)\mathrm{sinc}\left(\frac{y-y_m}{\Delta y}\right)\mathrm{sinc}\left(\frac{z-z_n}{\Delta z}\right)\\&+e_N(x,y,z,\Delta x,\Delta y,\Delta z).\end{aligned} \tag{3.25}$$

All the solutions obtained in chapter two contain one or more convolutions between a kernel and a function of the electric and magnetic field. Without loss of generality such integrals can be written as

$$I=\iiint_{R(\hat{x},\hat{y},\hat{z})} H(x-\hat{x},y-\hat{y},z-\hat{z},t)\,g(\hat{x},\hat{y},\hat{z})\,d\hat{x}\,d\hat{y}\,d\hat{z}. \tag{3.26}$$

Substituting the function g with its sampled representation, given by equation (3.25), yields

$$\begin{aligned}I=&\iiint_{R(\hat{x},\hat{y},\hat{z})} H(x-\hat{x},y-\hat{y},z-\hat{z},t)\\&\sum_{l=-N}^{N}\sum_{m=-N}^{N}\sum_{n=-N}^{N} g(x_l,y_m,z_n)\,\mathrm{sinc}\left(\frac{\hat{x}-x_l}{\Delta x}\right)\mathrm{sinc}\left(\frac{\hat{y}-y_m}{\Delta y}\right)\mathrm{sinc}\left(\frac{\hat{z}-z_n}{\Delta z}\right)d\hat{x}\,d\hat{y}\,d\hat{z}\\&+\iiint_{R(\hat{x},\hat{y},\hat{z})} H(x-\hat{x},y-\hat{y},z-\hat{z},t)\,e_N(\hat{x},\hat{y},\hat{z},\Delta x,\Delta y,\Delta z)\,d\hat{x}\,d\hat{y}\,d\hat{z}.\end{aligned} \tag{3.27}$$

Since the limits of the sum are independent of the limits on the integrals their order can be exchanged producing

$$I(x,y,z,t)=\sum_{l=-N}^{N}\sum_{m=-N}^{N}\sum_{n=-N}^{N}S(x,y,z,x_l,y_m,z_n,t)\,g(x_l,y_m,z_n)+\varepsilon(x,y,z,\Delta x,\Delta y,\Delta z), \quad (3.28)$$

where the summation kernel $S(x,y,z,x_l,y_m,z_n,t)$ is

$$\begin{aligned}&S(x,y,z,x_l,y_m,z_n,t)=\\&\iiint_{R(\hat{x},\hat{y},\hat{z})}H(x-\hat{x},y-\hat{y},z-\hat{z},t)\operatorname{sinc}\left(\frac{\hat{x}-x_l}{\Delta x}\right)\operatorname{sinc}\left(\frac{\hat{y}-y_m}{\Delta y}\right)\operatorname{sinc}\left(\frac{\hat{z}-z_n}{\Delta z}\right)d\hat{x}\,d\hat{y}\,d\hat{z},\end{aligned} \quad (3.29)$$

and the error term $\varepsilon(x,y,z,\Delta x,\Delta y,\Delta z)$ is given by

$$\varepsilon(x,y,z,\Delta x,\Delta y,\Delta z)=\iiint_{R(\hat{x},\hat{y},\hat{z})}H(x-\hat{x},y-\hat{y},z-\hat{z},t)\,e_N(\hat{x},\hat{y},\hat{z},\Delta x,\Delta y,\Delta z)\,d\hat{x}\,d\hat{y}\,d\hat{z}. \quad (3.30)$$

Therefore, if the kernel $S(x,y,z,x_l,y_m,z_n,t)$ is known, equation (3.28) can be used to compute the integral $I(x,y,z,t)$. The same reasoning applies to the two-dimensional case.

The three-dimensional kernel corresponding to the function $\delta(vt-r)/2r$ is computed below:

$$S_2\left(x,y,z,x_l,y_m,z_n,t\right)=$$
$$\iiint\limits_{R(\hat{x},\hat{y},\hat{z})} \frac{\delta\left(vt-r\right)}{2r}\operatorname{sinc}\left(\frac{\hat{x}-x_l}{\Delta x}\right)\operatorname{sinc}\left(\frac{\hat{y}-y_m}{\Delta y}\right)\operatorname{sinc}\left(\frac{\hat{z}-z_n}{\Delta z}\right)d\hat{x}\,d\hat{y}\,d\hat{z}, \tag{3.31}$$

where $r^2=\left(x-\hat{x}\right)^2+\left(y-\hat{y}\right)^2+\left(z-\hat{z}\right)^2$. Using spherical polar angular coordinates θ,ϕ with $\left(x,y,z\right)$ as the origin the equation can be written as

$$\begin{aligned}&S_2\left(x,y,z,x_l,y_m,z_n,t\right)=\\&\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\delta\left(vt-R\right)}{2}\operatorname{sinc}\left(\frac{x-x_l+R\sin\theta\cos\phi}{\Delta x}\right)\operatorname{sinc}\left(\frac{y-y_m+R\sin\theta\sin\phi}{\Delta y}\right)\\&\quad\operatorname{sinc}\left(\frac{z-z_n+R\cos\theta}{\Delta z}\right)R\sin\theta\,dR\,d\theta\,d\phi.\end{aligned} \tag{3.32}$$

Using properties of the delta symbol the expression can be simplified to

$$\begin{aligned}&S_2\left(x,y,z,x_l,y_m,z_n,t\right)=\\&\frac{vt}{2}\int_0^{2\pi}\int_0^{\pi}\operatorname{sinc}\left(\frac{x-x_l+vt\sin\theta\cos\phi}{\Delta x}\right)\operatorname{sinc}\left(\frac{y-y_m+vt\sin\theta\sin\phi}{\Delta y}\right)\\&\qquad\operatorname{sinc}\left(\frac{z-z_n+vt\cos\theta}{\Delta z}\right)\sin\theta\,d\theta\,d\phi.\end{aligned} \tag{3.33}$$

Using a similar approach, the three-dimensional kernel corresponding to the function $\pi u\left(vt-r\right)/r$ is found to be

$$\begin{aligned}&S_1\left(x,y,z,x_l,y_m,z_n,t\right)=\\&\pi\int_0^{2\pi}\int_0^{\pi}\int_0^{vt}R\operatorname{sinc}\left(\frac{x-x_l+R\sin\theta\cos\phi}{\Delta x}\right)\operatorname{sinc}\left(\frac{y-y_m+R\sin\theta\sin\phi}{\Delta y}\right)\\&\qquad\operatorname{sinc}\left(\frac{z-z_n+R\cos\theta}{\Delta z}\right)\sin\theta\,dR\,d\theta\,d\phi.\end{aligned} \tag{3.34}$$

The following is obtained by letting $\Delta x = \Delta y = \Delta z = vt = \Delta$ and computing the kernels at location (x_i, y_j, z_k):

$$S_1(x_i, y_j, z_k, x_l, y_m, z_n, t) =$$
$$\pi \int_0^{2\pi}\int_0^{\pi}\int_0^{\Delta} R \,\mathrm{sinc}\left(i - l + \frac{R\sin\theta\cos\phi}{\Delta}\right)\mathrm{sinc}\left(j - m + \frac{R\sin\theta\sin\phi}{\Delta}\right) \tag{3.35}$$
$$\mathrm{sinc}\left(k - n + \frac{R\cos\theta}{\Delta}\right)\sin\theta \, dR\, d\theta\, d\phi$$

and

$$S_2(x_i, y_j, z_k, x_l, y_m, z_n, t) =$$
$$\frac{\Delta}{2}\int_0^{2\pi}\int_0^{\pi} \mathrm{sinc}(i - l + \sin\theta\cos\phi)\,\mathrm{sinc}(j - m + \sin\theta\sin\phi) \tag{3.36}$$
$$\mathrm{sinc}(k - n + \cos\theta)\sin\theta\, d\theta\, d\phi$$

Equations (3.35) and (3.36) are the kernels needed for the three-dimensional case. Of course, in order for this to work, the wavenumbers must be restricted to the range $|k| \le 1/(2\Delta)$.

3.4 Absorbing Boundary Conditions

Although in this chapter discrete solutions have been given only for the one-dimensional case, it is clear that any extension to higher dimensions will necessitate special boundary conditions. Unlike the one-dimensional case, where it is possible to match the impedance of a medium and remove all reflections; in higher dimensions absorbing boundary conditions (ABCs) are required. The most effective ABC is the Perfectly Matched Layer

(PML) due to Berenger [5], an absorbing medium that surrounds the region of interest. This medium absorbs all incoming waves and heavily attenuates them as they travel though it. The absorbing medium has finite dimensions, and is generally terminated with a perfect electric conductor (PEC). A reflection does occur at this junction, however, due to the attenuation the waves experience as they travel to and form the PEC, by the time the waves do return to the medium of interest, their energy is negligible.

3.5 Software

The following MATLAB programs are used to generate the graphs in this chapter.

```
function [vk, ik] = tline_demo1 (L, C, RG, RL, len, n, iter, k);

% This function simulates a losless transmission line attached to a load and a generator.
% The input parameters are
%
%   L    =  inductance per unit length (in Henrys/meter)
%   C    =  capacitance per unit length (in Farads/meter)
%   RG   =  generator resistance (in Ohms)
%   RL   =  load resistance (in Ohms)
%   len  =  length of the tline (in meters)
%   n    =  number of partitions of the tline (j=1,2,...,n)
%   iter =  total number of iterations
%   k    =  specifies the location where V and I are recorded and returned as the output
%           of the function
%
%   usage:
%
%    [vk, ik, dt] = tline_demo1 (0.5*10^-6, 200*10^-12, 25, 75, 10^4, 100, 1200, 1);
%

% constants
v=1/sqrt(L*C);
Z0=sqrt(L/C);
dx=len/n;
dt=dt/v;
pG=(RG-Z0)/(RG+Z0)
pL=(RL-Z0)/(RL+Z0)

% arrays
V=zeros(1,n);
I=zeros(1,n);
Vt=zeros(1,n);
It=zeros(1,n);
```

```
Vh=[];
Ih=[];

%%%%%%%%%%% Main algorithm %%%%%%%%%%%%%
for j=1:iter

Vg=10*sin(2*pi*f0*dt*(j-1));

% update inner locations (k=2,...,n-1)

Vt(2:n-1) = (1/2)*[V(3:n) + V(1:n-2)] - (Z0/2)*[I(3:n) - I(1:n-2)];
It(2:n-1) = -(1/2/Z0)*[V(3:n) - V(1:n-2)] + (1/2)*[I(3:n) + I(1:n-2)];

% update at the generator (k=1)

Vt(1) = [(1/2)*V(2) - (Z0/2)*I(2)]*(1+pG) + Z0*Vg/(Z0+RG);
It(1) = [(-1/2/Z0)*V(2) + (1/2)*I(2)]*(1-pG) + Vg/(Z0+RG);

% update at the load (k=n)

Vt(n) = [(1/2)*V(n-1) + (Z0/2)*I(n-1)]*(1+pL);
It(n) = [(1/2/Z0)*V(n-1) + (1/2)*I(n-1)]*(1-pL);

% move from temporary location to working array
V=Vt;
I=It;

% store history
Vh=[Vh Vt(k)];
Ih=[Ih It(k)];

end

vk=Vh;
ik=Ih;


function [vk, ik, dt] = tline_demo2 (L, C, RG, RL, f0, fs, len, n, iter, k);

% This function simulates a losless transmission line attached to a load and a generator.
% The generator produces a sinusoid of frequency f0
%
% The input parameters are
%
%   L    =  inductance per unit length (in Henrys/meter)
%   C    =  capacitance per unit length (in Farads/meter)
%   RG   =  generator resistance (in Ohms)
%   RL   =  load resistance (in Ohms)
%...f0   =  frequency of the sinusoid produced by the generator
%   fs   =  sampling frequency
%   len  =  length of the tline (in meters)
%   n    =  number of partitions of the tline (j=1,2,...,n)
%   iter =  total number of iterations
%   k    =  specifies the location where V and I are recorded and returned as the output
%           of the function
%
%   usage:
%
%    [vk, ik, dt] =
%         tline_demo2 (0.5*10^-6, 200*10^-12, 25, 75, 1e5, 4e6, 10^4, 100, 1200, 1);
%

% constants
v=1/sqrt(L*C);
Z0=sqrt(L/C);
dt=1/fs;
```

```
dx=v*dt;
pG=(RG-Z0)/(RG+Z0)
pL=(RL-Z0)/(RL+Z0)

% arrays
V=zeros(1,n);
I=zeros(1,n);
Vt=zeros(1,n);
It=zeros(1,n);
Vh=[];
Ih=[];

%%%%%%%%%%% Main algorithm %%%%%%%%%%%%%
for j=1:iter

Vg=10;

% update inner locations (k=2,...,n-1)

Vt(2:n-1) = (1/2)*[V(3:n) + V(1:n-2)] - (Z0/2)*[I(3:n) - I(1:n-2)];
It(2:n-1) = -(1/2/Z0)*[V(3:n) - V(1:n-2)] + (1/2)*[I(3:n) + I(1:n-2)];

% update at the generator (k=1)

Vt(1) = [(1/2)*V(2) - (Z0/2)*I(2)]*(1+pG) + Z0*Vg/(Z0+RG);
It(1) = [(-1/2/Z0)*V(2) + (1/2)*I(2)]*(1-pG) + Vg/(Z0+RG);

% update at the load (k=n)

Vt(n) = [(1/2)*V(n-1) + (Z0/2)*I(n-1)]*(1+pL);
It(n) = [(1/2/Z0)*V(n-1) + (1/2)*I(n-1)]*(1-pL);


% move from temporary location to working array
V=Vt;
I=It;

% store history
Vh=[Vh Vt(k)];
Ih=[Ih It(k)];

end

vk=Vh;
ik=Ih;



function [] = tline2 (L1, C1, L2, C2, RG, RL, len1, len2, n, m, iter);

% This function simulates a pulse traveling through two cascaded transmission lines
% The input parameters are
%
%   L1   =  inductance per unit length (in Henrys/meter) of the 1st tline
%   C1   =  capacitance per unit length (in Farads/meter) of the 1st tline
%   L2   =  inductance per unit length (in Henrys/meter) of the 2nd tline
%   C2   =  capacitance per unit length (in Farads/meter) of the 2nd tline
%   RG   =  generator resistance (in Ohms)
%   RL   =  load resistance (in Ohms)
%   len1 =  length of the 1st tline
%   len2 =  length of the 2nd tline
%   n    =  number of partitions of the 1st tline (j=1,2,...,n)
%   m    =  number of partitions of the 2nd tline (j=1,2,...,m)
%   iter =  total number of iterations
%
%   ex:
```

```
%
%    tline2 (0.5*10^-6, 200*10^-12, 0.5*10^-6, (200*10^-12)/4, 50, 50, 1e3, 3e3, 400,
800, 2200, 1);
%


% constants
v1=1/sqrt(L1*C1)
Z1=sqrt(L1/C1)
v2=1/sqrt(L2*C2)
Z2=sqrt(L2/C2)

dx1=len1/n;
dt1=dx1/v1;
dx2=len2/m;
dt2=dx2/v2;

f0=1e5;
fs=4e6;
pG=(RG-Z1)/(RG+Z1)
pL=(RL-Z2)/(RL+Z2)

R0=50;
Z2p=Z2*R0/(Z2+R0)
Z1p=Z1*R0/(Z1+R0)

p12=(Z2p-Z1)/(Z1+Z2p)
p21=(Z1p-Z2)/(Z1p+Z2)

% arrays
V1=zeros(1,n);
I1=zeros(1,n);
Vt1=zeros(1,n);
It1=zeros(1,n);
V2=zeros(1,m);
I2=zeros(1,m);
Vt2=zeros(1,m);
It2=zeros(1,m);

%%%%%%%%%%%% Main algorithm %%%%%%%%%%%%%%
for j=1:iter

% generate pulse
if (j<=50)
   Vg=10;
else
   Vg=0;
end;


% update inner locations (k=2,...,n-1) of first transmission line

Vt1(2:n-1) = (1/2)*[V1(3:n) + V1(1:n-2)] - (Z1/2)*[I1(3:n) - I1(1:n-2)];
It1(2:n-1) = -(1/2/Z1)*[V1(3:n) - V1(1:n-2)] + (1/2)*[I1(3:n) + I1(1:n-2)];

% update at the generator (k=1)

Vt1(1) = [(1/2)*V1(2) - (Z1/2)*I1(2)]*(1+pG) + Z1*Vg/(Z1+RG);
It1(1) = [(-1/2/Z1)*V1(2) + (1/2)*I1(2)]*(1-pG) + Vg/(Z1+RG);

% update at the boundary

Vt1(n) = [(1/2)*V1(n-1)+ (Z1/2)*I1(n-1)] + [(1/2)*V2(1) - (Z1/2)*I2(1)];
It1(n) = [(1/2/Z1)*V1(n-1)+ (1/2)*I1(n-1)] + [(-1/2/Z1)*V2(1) + (1/2)*I2(1)];

Vt2(1) = [(1/2)*V1(n)+ (Z2/2)*I1(n)] + [(1/2)*V2(2) - (Z2/2)*I2(2)];
It2(1) = [(1/2/Z2)*V1(n) + (1/2)*I1(n)] + [(-1/2/Z2)*V2(2) + (1/2)*I2(2)];
```

```
% update inner locations (k=2,...,m-1) of the second transmission line

Vt2(2:m-1) = (1/2)*[V2(3:m) + V2(1:m-2)] - (Z2/2)*[I2(3:m) - I2(1:m-2)];
It2(2:m-1) = -(1/2/Z2)*[V2(3:m) - V2(1:m-2)] + (1/2)*[I2(3:m) + I2(1:m-2)];

% update at the load (k=m)

Vt2(m) = [(1/2)*V2(m-1) + (Z2/2)*I2(m-1)]*(1+pL);
It2(m) = [(1/2/Z2)*V2(m-1) + (1/2)*I2(m-1)]*(1-pL);

% adjust V and I for second transmission line

xi = 1:v1/v2:length(V2);
V2x = interp1(V2,xi);
I2x = interp1(I2,xi);

% move from temporary location to working array
V1=Vt1;
I1=It1;
V2=Vt2;
I2=It2;

% adjust V and I for second transmission line

xi = 1:v1/v2:length(V2);
V2x = interp1(V2,xi);
I2x = interp1(I2,xi);

end




function [] = wave1d2 (iter);

% This function simulates a pulse propagating through a medium made up of
% two regions of different permitivities glued to one another
%
% The input parameters are
%
%   iter =  total number of iterations
%
%   ex: [E, H] = wave1d2 (0.5*10^-6, 200*10^-12, 25, 75, 10^4, 100, 1200, 1);
%


n1=400;
n2=600;

ep1=11*8.854e-12;
mu1=pi*4e-7;
ep2=1.7*8.854e-12;
mu2=pi*4e-7;

% constants
v1=1/sqrt(mu1*ep1);
nu1=sqrt(mu1/ep1);
dt1=.001;
dx1=v1*dt1;

v2=1/sqrt(mu2*ep2);
nu2=sqrt(mu2/ep2);
dt2=.001;
dx2=v2*dt2;
```

```
% arrays
E1=zeros(1,n1);
H1=zeros(1,n1);
Et1=zeros(1,n1);
Ht1=zeros(1,n1);

E2=zeros(1,n2);
H2=zeros(1,n2);
Et2=zeros(1,n2);
Ht2=zeros(1,n2);

% forcing function parameters
beta=110;
E0=100;

%%%%%%%%%%% Main algorithm %%%%%%%%%%%%%
for j=1:iter

if (j <= 2*beta)
   xn=(1-j/beta).^2;
   Eg=E0*exp(-16.*xn);
else
   Eg=0;
end

% update at the generator (k=1)
Et1(1) = [(1/2)*E1(2) - (nu1/2)*H1(2)] + Eg/2;
Ht1(1) = [(-1/2/nu1)*E1(2) + (1/2)*H1(2)] + Eg/2/nu1;

% update inner locations (k=2,...,n-1)
[Et,Ht]=update_inner(E1,H1,nu1,n1);
Et1(2:n1-1)=Et;
Ht1(2:n1-1)=Ht;

[Et,Ht]=update_inner(E2,H2,nu2,n2);
Et2(2:n2-1)=Et;
Ht2(2:n2-1)=Ht;

% update boundaries
[Ea,Ha,Eb,Hb]=update_boundary(E1(n1-2),H1(n1-2),E1(n1-1),H1(n1-
1),nu1,E2(1),H2(1),E2(2),H2(2),nu2);
Et1(n1)=Ea;
Ht1(n1)=Ha;
Et2(1)=Eb;
Ht2(1)=Hb;

% update at the load (k=n) using the algorithm:
Et2(n2) = [(1/2)*E2(n2-1) + (nu2/2)*H2(n2-1)];
Ht2(n2) = [(1/2/nu2)*E2(n2-1) + (1/2)*H2(n2-1)];

xi = 1:v1/v2:floor(length(E2)*v1/v2);
E2x = interp1(E2,xi);
H2x = interp1(H2,xi);

% move from temporary location to working array
E1=Et1;
H1=Ht1;
E2=Et2;
H2=Ht2;

xi = 1:v1/v2:floor(length(E2)*v1/v2);
E2x = interp1(E2,xi);
H2x = interp1(H2,xi);

end
```

```
function [Et,Ht]=update_inner(E,H,nu,n)

% update inner locations (k=2,...,n-1) using the algorithm:

Et = (1/2)*[E(3:n) + E(1:n-2)] - (nu/2)*[H(3:n) - H(1:n-2)];
Ht = -(1/2/nu)*[E(3:n) - E(1:n-2)] + (1/2)*[H(3:n) + H(1:n-2)];



function [Eta,Hta,Etb,Htb]=update_boundary(Ea1,Ha1,Ea2,Ha2,nu1,Eb1,Hb1,Eb2,Hb2,nu2);

% update at the boundary

Eta = [(1/2)*Ea1 + (nu1/2)*Ha1] + [(1/2)*Eb1 - (nu1/2)*Hb1];
Hta = [(1/2/nu1)*Ea1 + (1/2)*Ha1] + [(-1/2/nu1)*Eb1 + (1/2)*Hb1];

Etb = [(1/2)*Ea2 + (nu2/2)*Ha2] + [(1/2)*Eb2 - (nu2/2)*Hb2];
Htb = [(1/2/nu2)*Ea2 + (1/2)*Ha2] + [(-1/2/nu2)*Eb2 + (1/2)*Hb2];
```

Chapter 4

Conclusion and Future Work

In this thesis a novel mathematical approach was used to obtain analytical solutions to Maxwell's equations in a homogeneous medium. The gist of the method was the transformation of the system of coupled partial differential equations into a system of coupled ordinary differential equations, by making use of the Fourier transform. Exact solutions to Maxwell's equations in a lossless homogeneous medium in one, two, and three dimensions were presented. The method was also used to obtain solutions to the one-dimensional transmission line equation.

The one-dimensional solutions were then converted from continuous to discrete. Programs implementing the discrete formulas were used to simulate a series of cases and the results were consistent with the theory. A technique for obtaining discrete solutions for the two- and three-dimensional cases was then discussed, but no solutions were given for the higher dimensions.

The kernel matrices for the lossy case are not given, because their sheer complexity makes them too cumbersome for inclusion. Future work should attempt to simplify these expressions and find a pattern to the lossy kernels, as was done for the lossless case. Future work should also try to produce numerical solutions for the two- and three-dimensional cases.

The mathematical method presented in this document to solve Maxwell's equations is quite broad in scope. In general, it can be applied to solve systems of partial differential equations with constant coefficients. The system of Maxwell's equations is, to be precise, just a specific case among an infinite number of them. Future work should exploit the generality of this method by analyzing and solving other systems of this kind.

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APPENDIX A

The Fourier Transform and Some of Its Properties

In this document the following definitions for the Fourier transform and its inverse are used, as given in [2]:

$$\mathbf{G}(\mathbf{k},t)=\int_{\mathbf{x}}\mathbf{g}(\mathbf{x},t)e^{-i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \tag{A.1}$$

and

$$\mathbf{g}(\mathbf{x},t)=\int_{\mathbf{x}}\mathbf{G}(\mathbf{k},t)e^{i2\pi\mathbf{k}\cdot\mathbf{x}}d\mathbf{k} \tag{A.2}$$

In equations $(A.1)$ and $(A.2)$ the integral is in symbolic form. The formulas for one-, two-, and three-dimensional cases in explicit form are given by

$$\begin{aligned}
\mathbf{G}(p,t)&=\int_{-\infty}^{\infty}\mathbf{g}(x,t)e^{-i2\pi px}dx\\
\mathbf{G}(p,q,t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}(x,y,t)e^{-i2\pi(px+qy)}dx\,dy\\
\mathbf{G}(p,q,r,t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}(x,y,z,t)e^{-i2\pi(px+qy+rz)}dx\,dy\,dz
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
\mathbf{g}(x,t)&=\int_{-\infty}^{\infty}\mathbf{G}(p,t)e^{i2\pi px}dp\\
\mathbf{g}(x,y,t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(p,q,t)e^{i2\pi(px+qy)}dp\,dq\\
\mathbf{g}(x,y,z,t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{G}(p,q,r,t)e^{i2\pi(px+qy+rz)}dp\,dq\,dr
\end{aligned} \tag{A.4}$$

Some of the properties of the Fourier transform used in this document are summarized in the following tables

Theorem	$f(x)$	$F(p)$
Addition	$f(x)+g(x)$	$F(p)+G(p)$
Shift	$f(x-a)$	$e^{-i2\pi ap}F(p)$
Multiplication	$f(x)g(x)$	$F(p)*G(p)$
Derivative	$f'(x)$	$i2\pi pF(p)$

Theorem	$f(x,y)$	$F(p,q)$
Addition	$f(x,y)+g(x,y)$	$F(p)+G(p)$
Shift	$f(x-a,y-b)$	$e^{-i2\pi(ap+bq)}F(p,q)$
Multiplication	$f(x,y)g(x,y)$	$F(p,q)*G(p,q)$
Derivative	$\frac{\partial}{\partial x}f(x,y)$	$i2\pi pF(p,q)$
Derivative	$\frac{\partial}{\partial y}f(x,y)$	$i2\pi qF(p,q)$

Theorem	$f(x,y,z)$	$F(p,q,r)$
Addition	$f(x,y,z)+g(x,y,z)$	$F(p,q,r)+G(p,q,r)$
Shift	$f(x-a,y-b,z-c)$	$e^{-i2\pi(ap+bq+cr)}F(p,q,r)$
Multiplication	$f(x,y,z)g(x,y,z)$	$F(p,q,r)*G(p,q,r)$

Derivative	$\frac{\partial}{\partial x} f(x,y,z)$	$i2\pi pF(p,q,r)$
Derivative	$\frac{\partial}{\partial y} f(x,y,z)$	$i2\pi qF(p,q,r)$
Derivative	$\frac{\partial}{\partial z} f(x,y,z)$	$i2\pi rF(p,q,r)$

APPENDIX B

The Kernel Matrix

The exponential matrix $\mathbf{H}(\mathbf{k},t)=e^{\mathbf{P}(\mathbf{k})t}$ is itself a matrix and can be written as [3]

$$\mathbf{H}(\mathbf{k},t)=e^{\mathbf{P}(\mathbf{k})t}=\mathbf{M}(\mathbf{k})\mathbf{D}\left(e^{\lambda_n t}\right)\mathbf{M}^{-1}(\mathbf{k}) \tag{B.1}$$

where $\mathbf{M}(\mathbf{k})$ is a modal matrix whose columns are the eigenvectors of $\mathbf{P}(\mathbf{k})$ and $\mathbf{D}\left(e^{\lambda_n t}\right)$ is a diagonal matrix whose elements are the exponentials $e^{\lambda_n t}$ where the λ_n's are the eigenvalues of $\mathbf{P}(\mathbf{k})$.

B.1 One Dimensional Case

In the case of a one-dimensional homogeneous medium where ε, μ, σ_e, and σ_m, are constants, the matrix $\mathbf{P}(p)$ is given by

$$\mathbf{P}(p)=\begin{bmatrix} -\dfrac{\sigma_e}{\varepsilon} & -\dfrac{i2\pi p}{\varepsilon} \\ -\dfrac{i2\pi p}{\mu} & -\dfrac{\sigma_m}{\mu} \end{bmatrix} \tag{B.2}$$

The eigenvalues of $\mathbf{P}(p)$ and its corresponding eigenvectors are

$$\lambda_{1,2}=-\left(\frac{\sigma_e}{2\varepsilon}+\frac{\sigma_m}{2\mu}\right)\pm\sqrt{\left(\frac{\sigma_e}{2\varepsilon}-\frac{\sigma_m}{2\mu}\right)^2+\frac{(2\pi p)^2}{\mu\varepsilon}} \tag{B.3}$$

$$\mathbf{v}_{1,2} = a\begin{pmatrix} 1 \\ \dfrac{i\varepsilon}{2\pi p}\left(\lambda_{1,2} + \dfrac{\sigma_e}{\varepsilon}\right) \end{pmatrix} \tag{B.4}$$

where a is a constant.

In the special case of a one-dimensional homogeneous medium where $\sigma_e = 0$, $\sigma_m = 0$, and ε and μ are constants, the matrix $\mathbf{P}(p)$ simplifies to

$$\mathbf{P}(p) = \begin{bmatrix} 0 & -\dfrac{i2\pi p}{\varepsilon} \\ -\dfrac{i2\pi p}{\mu} & 0 \end{bmatrix} \tag{B.5}$$

In this case the eigenvalues of $\mathbf{P}(p)$ and its corresponding eigenvectors are

$$\lambda_{1,2} = \pm i2\pi p v \tag{B.6}$$

$$\mathbf{v}_1 = a\begin{bmatrix} 1 \\ 1/\eta \end{bmatrix} \qquad \mathbf{v}_2 = a\begin{bmatrix} 1 \\ -1/\eta \end{bmatrix} \tag{B.7}$$

where $v = 1/\sqrt{\varepsilon\mu}$, and $\eta = \sqrt{\varepsilon/\mu}$. The matrix kernel $\mathbf{H}(p,t)$ in this case is given by

$$\mathbf{H}(p,t) = e^{\mathbf{P}(p)t} = \begin{bmatrix} \cos(2\pi p v t) & -i\eta\sin(2\pi p v t) \\ -\dfrac{i}{\eta}\sin(2\pi p v t) & \cos(2\pi p v t) \end{bmatrix} \tag{B.8}$$

B.2 Two-dimensional case

For the two dimensional case, the matrix $\mathbf{P}(p,q)$ is now

$$\mathbf{P}(p,q)=\begin{bmatrix} -\frac{\sigma_e}{\varepsilon} & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} \\ -\frac{i2\pi q}{\mu} & -\frac{\sigma_m}{\mu} & 0 \\ \frac{i2\pi p}{\mu} & 0 & -\frac{\sigma_m}{\mu} \end{bmatrix} \tag{B.9}$$

For this case obtaining the eigenvalues, eigenvectors, and the exponential matrix necessitates the use of specialized mathematical software. Since these are matrices whose entries are algebraic rather than numeric, the need is for symbolic mathematical software. The symbolic software MuPAD Pro 3.1 is used to assist in the computation of the kernel matrices, but even with the aid of this software the results obtained for the matrix in $(\mathrm{B}.9)$ are quite unwieldy. To obtain a manageable kernel matrix, only the lossless medium is considered. For this case the matrix $\mathbf{P}(p,q)$ is

$$\mathbf{P}(p,q)=\begin{bmatrix} 0 & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} \\ -\frac{i2\pi q}{\mu} & 0 & 0 \\ \frac{i2\pi p}{\mu} & 0 & 0 \end{bmatrix} \tag{B.10}$$

The results from MuPAD Pro 3.1 are given below

- `A := matrix([[0,-I*2*pi*q/eps,I*2*pi*p/eps], [-I*2*pi*q/mu,0,0], [I*2*pi*p/mu,0,0]])`

$$\begin{pmatrix} 0 & \frac{(-2\cdot \mathrm{i})\cdot \mathrm{pi}\cdot q}{\mathrm{eps}} & \frac{(2\cdot \mathrm{i})\cdot p\cdot \mathrm{pi}}{\mathrm{eps}} \\ \frac{(-2\cdot \mathrm{i})\cdot \mathrm{pi}\cdot q}{\mathrm{mu}} & 0 & 0 \\ \frac{(2\cdot \mathrm{i})\cdot p\cdot \mathrm{pi}}{\mathrm{mu}} & 0 & 0 \end{pmatrix}$$

- `linalg::eigenvalues(A)`

$$\left\{0,\ -2\cdot \mathrm{pi}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}},\ 2\cdot \mathrm{pi}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}}\right\}$$

- `linalg::eigenvectors(A)`

$$\left[\left[-2\cdot\mathrm{pi}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}},1,\left[\begin{pmatrix}\frac{\mathrm{i}\cdot\mathrm{mu}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}}}{p}\\ -\frac{q}{p}\\ 1\end{pmatrix}\right]\right],\left[2\cdot\mathrm{pi}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}},1,\left[\begin{pmatrix}\frac{(-1\cdot\mathrm{i})\cdot\mathrm{mu}\cdot\sqrt{-\frac{p^2+q^2}{\mathrm{eps}\cdot\mathrm{mu}}}}{p}\\ -\frac{q}{p}\\ 1\end{pmatrix}\right]\right],\left[0,1,\left[\begin{pmatrix}0\\ \frac{p}{q}\\ 1\end{pmatrix}\right]\right]\right]$$

- `y:=exp(A*t);`
- `y[1..3,1]`

$$\begin{pmatrix}\frac{e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2}+\frac{e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2}\\ \frac{\mathrm{i}\cdot q\cdot e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}{\mathrm{mu}\cdot(2\cdot p^2+2\cdot q^2)}-\frac{\mathrm{i}\cdot q\cdot e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}{\mathrm{mu}\cdot(2\cdot p^2+2\cdot q^2)}\\ \frac{\mathrm{i}\cdot p\cdot e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}{\mathrm{mu}\cdot(2\cdot p^2+2\cdot q^2)}-\frac{\mathrm{i}\cdot p\cdot e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}{\mathrm{mu}\cdot(2\cdot p^2+2\cdot q^2)}\end{pmatrix}$$

- `y[1..3,2]`

$$\begin{pmatrix}\frac{\left(\frac{1}{2}\cdot\mathrm{i}\right)\cdot\mathrm{mu}\cdot q\cdot e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}-\frac{\left(\frac{1}{2}\cdot\mathrm{i}\right)\cdot\mathrm{mu}\cdot q\cdot e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}}\\ \frac{p^2}{p^2+q^2}+\frac{q^2\cdot e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2}+\frac{q^2\cdot e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2}\\ \frac{p\cdot q}{p^2+q^2}-\frac{p\cdot q\cdot e^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2}-\frac{p\cdot q\cdot e^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot(p^2+q^2)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2}\end{pmatrix}$$

• y[1..3,3]

$$\begin{pmatrix} \dfrac{\left(\frac{1}{2}\cdot\mathrm{i}\right)\cdot\mathrm{mu}\cdot p\cdot \mathrm{e}^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}} - \dfrac{\left(\frac{1}{2}\cdot\mathrm{i}\right)\cdot\mathrm{mu}\cdot p\cdot \mathrm{e}^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot p^2-\mathrm{eps}\cdot\mathrm{mu}\cdot q^2}} \\ \dfrac{p\cdot q}{p^2+q^2} - \dfrac{p\cdot q\cdot \mathrm{e}^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2} - \dfrac{p\cdot q\cdot \mathrm{e}^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2} \\ \dfrac{q^2}{p^2+q^2} + \dfrac{p^2\cdot \mathrm{e}^{-\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2} + \dfrac{p^2\cdot \mathrm{e}^{\frac{2\cdot\mathrm{pi}\cdot t\cdot\sqrt{-\mathrm{eps}\cdot\mathrm{mu}\cdot\left(p^2+q^2\right)}}{\mathrm{eps}\cdot\mathrm{mu}}}}{2\cdot p^2+2\cdot q^2} \end{pmatrix}$$

After some algebra, the kernel matrix $\mathbf{H}(p,q,t)$ simplifies to

$$\mathbf{H}(p,q,t) = \begin{bmatrix} 1-k^2F & -i\eta qG & i\eta pG \\ \dfrac{-iq}{\eta}G & 1-q^2F & pqF \\ \dfrac{ip}{\eta}G & pqF & 1-p^2F \end{bmatrix} \tag{B.11}$$

where $F=\left(1-\cos(2\pi k\upsilon t)\right)/k^2$, $G=\sin(2\pi k\upsilon t)/k$, $k=\sqrt{p^2+q^2}$, $\upsilon=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.

B.3 Three-Dimensional Case

For the three-dimensional case, the matrix $\mathbf{P}(p,q,r)$ is given by

$$\mathbf{P}(p,q,r)=\begin{bmatrix} -\frac{\sigma_e}{\varepsilon} & 0 & 0 & 0 & -\frac{i2\pi r}{\varepsilon} & \frac{i2\pi q}{\varepsilon} \\ 0 & -\frac{\sigma_e}{\varepsilon} & 0 & \frac{i2\pi r}{\varepsilon} & 0 & -\frac{i2\pi p}{\varepsilon} \\ 0 & 0 & -\frac{\sigma_e}{\varepsilon} & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} & 0 \\ 0 & \frac{i2\pi r}{\mu} & -\frac{i2\pi q}{\mu} & -\frac{\sigma_m}{\mu} & 0 & 0 \\ -\frac{i2\pi r}{\mu} & 0 & \frac{i2\pi p}{\mu} & 0 & -\frac{\sigma_m}{\mu} & 0 \\ \frac{i2\pi q}{\mu} & -\frac{i2\pi p}{\mu} & 0 & 0 & 0 & -\frac{\sigma_m}{\mu} \end{bmatrix} \tag{B.12}$$

As with the two-dimensional case, keeping all the terms results in a formidable expression. Hence, the medium is restricted to being lossless. In this case, the matrix becomes

$$\mathbf{P}(p,q,r)=\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{i2\pi r}{\varepsilon} & \frac{i2\pi q}{\varepsilon} \\ 0 & 0 & 0 & \frac{i2\pi r}{\varepsilon} & 0 & -\frac{i2\pi p}{\varepsilon} \\ 0 & 0 & 0 & -\frac{i2\pi q}{\varepsilon} & \frac{i2\pi p}{\varepsilon} & 0 \\ 0 & \frac{i2\pi r}{\mu} & -\frac{i2\pi q}{\mu} & 0 & 0 & 0 \\ -\frac{i2\pi r}{\mu} & 0 & \frac{i2\pi p}{\mu} & 0 & 0 & 0 \\ \frac{i2\pi q}{\mu} & -\frac{i2\pi p}{\mu} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.13}$$

Application of the software yields

```
• A := matrix([[0,0,0,0,-i*2*pi*r/ep,i*2*pi*q/ep],
  [0,0,0,i*2*pi*r/ep,0,-i*2*pi*p/ep],[0,0,0,-
  i*2*pi*q/ep,i*2*pi*p/ep,0],[0,i*2*pi*r/mu,-
  i*2*pi*q/mu,0,0,0],[-i*2*pi*r/mu,0,
  i*2*pi*p/mu,0,0,0],[i*2*pi*q/mu,-i*2*pi*p/mu,0,0,0]])
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{2 \cdot i \cdot \mathrm{pi} \cdot r}{\mathrm{ep}} & \frac{2 \cdot i \cdot \mathrm{pi} \cdot q}{\mathrm{ep}} \\ 0 & 0 & 0 & \frac{2 \cdot i \cdot \mathrm{pi} \cdot r}{\mathrm{ep}} & 0 & -\frac{2 \cdot i \cdot p \cdot \mathrm{pi}}{\mathrm{ep}} \\ 0 & 0 & 0 & -\frac{2 \cdot i \cdot \mathrm{pi} \cdot q}{\mathrm{ep}} & \frac{2 \cdot i \cdot p \cdot \mathrm{pi}}{\mathrm{ep}} & 0 \\ 0 & \frac{2 \cdot i \cdot \mathrm{pi} \cdot r}{\mathrm{mu}} & -\frac{2 \cdot i \cdot \mathrm{pi} \cdot q}{\mathrm{mu}} & 0 & 0 & 0 \\ -\frac{2 \cdot i \cdot \mathrm{pi} \cdot r}{\mathrm{mu}} & 0 & \frac{2 \cdot i \cdot p \cdot \mathrm{pi}}{\mathrm{mu}} & 0 & 0 & 0 \\ \frac{2 \cdot i \cdot \mathrm{pi} \cdot q}{\mathrm{mu}} & -\frac{2 \cdot i \cdot p \cdot \mathrm{pi}}{\mathrm{mu}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
• exp(A,t)

array(1..6, 1..6,
  (1, 1) = p^2/(p^2 + q^2 + r^2) + exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2)\
)^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2) +
exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 +\
 q^2 + r^2))^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2),
  (1, 2) = p*q/(p^2 + q^2 + r^2) - 1/2/p*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 \
+ r^2))^(1/2))/(mu*q^2 + mu*r^2)*(mu*q^3 + mu*q*r^2) -
1/2/p*exp(2/ep*i/mu*pi*t\
*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(mu*q^2 + mu*r^2)*(mu*q^3 +
mu*q*r^2) + 1/p*q\
*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(q^2 +
r^2)/(2*p^2 + 2*q^\
2 + 2*r^2) + 1/p*q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(q^2 + r\
^2)/(2*p^2 + 2*q^2 + 2*r^2),
  (1, 3) = p*r/(p^2 + q^2 + r^2) - 1/p/r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 \
+ r^2))^(1/2))*(p^2 + q^2)*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2) -
1/p/r*exp(2/ep\
*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(p^2 + q^2)*(q^2 +
r^2)/(2*p^2 + 2*\
q^2 + 2*r^2) + 1/2/p*q/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(\
mu*q^2 + mu*r^2)*(mu*q^3 + mu*q*r^2) +
1/2/p*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2\
 + q^2 + r^2))^(1/2))/(mu*q^2 + mu*r^2)*(mu*q^3 + mu*q*r^2),
```

```
  (1, 4) = 1/2/p/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(mu*q^3\
 + mu*q*r^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/2/p/r*exp(2/ep*i/mu*\
pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(mu*q^3 +
mu*q*r^2)/(ep*mu*p^2 + ep*mu*q^\
2 + ep*mu*r^2)^(1/2) - 1/ep/p*q/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2
+ q^2 + r^2))\
^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 +
ep*mu*q^2 + ep*mu*r^2)\
^(1/2) + 1/ep/p*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(q^2 + \
r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*\,
           q^2 + ep*mu*r^2)^(1/2)
  (1, 5) = 1/ep/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(q^2 + r\
^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2) - 1/2*q/r\
*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(mu*q^3 +
mu*q*r^2)/(q^2 \
+ r^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/ep/r*exp(2/ep*i/mu*pi*t*(e\
p*mu*(p^2 + q^2 + r^2))^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 +
2*r^2)*(ep*mu*p^2 +\
 ep*mu*q^2 + ep*mu*r^2)^(1/2) +
1/2*mu*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2\
 + r^2))^(1/2))/(mu*q^2 + mu*r^2)*(mu*q^3 + mu*q\,
           *r^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2)
  (1, 6) = 1/2*mu*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 +\
 mu*r^2)*(mu*q^3 + mu*q*r^2)/(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2) - 1/2*ex\
p(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(mu*q^3 +
mu*q*r^2)/(q^2 + r\
^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2),
  (2, 1) = p*q/(p^2 + q^2 + r^2) - p*q*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2) -
p*q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2\
 + r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2),
  (2, 2) = 1/2*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2)) + 1/2*exp(2\
/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2)) + q^2/(p^2 + q^2 +
r^2) - q^2*ex\
p(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 + 2*q^2
+ 2*r^2) - q^\
2*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 +
2*q^2 + 2*r^2),
```

```
  (2, 3) = q*r/(p^2 + q^2 + r^2) - 1/2*q/r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^\
2 + r^2))^(1/2)) - 1/2*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2)) \
+ q/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(p^2 +
q^2)/(2*p^2 +\
 2*q^2 + 2*r^2) + q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(p^2 \
+ q^2)/(2*p^2 + 2*q^2 + 2*r^2),
  (2, 4) = 1/2/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(mu*q^2 + \
mu*r^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) - 1/2/r*exp(-
2/ep*i/mu*pi*t*(\
ep*mu*(p^2 + q^2 + r^2))^(1/2))*(mu*q^2 + mu*r^2)/(ep*mu*p^2 +
ep*mu*q^2 + ep*m\
u*r^2)^(1/2) + 1/ep*q^2/r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/\
(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2)
- 1/ep*q^2/r*\
exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 +
2*q^2 + 2*r^2)*(ep\
*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2),
  (2, 5) = 1/ep*p*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(2*p^\
2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/ep*p*q/r*exp(-\
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 + 2*q^2 +
2*r^2)*(ep*mu*\
p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/2*mu*p*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^\
2 + q^2 + r^2))^(1/2))/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2)
+ 1/2*p*q/r*ex\
p(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(mu*q^2 +
mu*r^2)/(q^2 + r^2\
)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2),
  (2, 6) = 1/2*p*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(mu*q^2 +\
 mu*r^2)/(q^2 + r^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/2*mu*p*exp(2\
/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(ep*mu*p^2 +
ep*mu*q^2 + ep*mu*r\
^2)^(1/2),
  (3, 1) = p*r/(p^2 + q^2 + r^2) - p*r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2) -
p*r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2\
 + r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2),
  (3, 2) = q*r/(p^2 + q^2 + r^2) - q*r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
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r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2) -
q*r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2\
 + r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2),
  (3, 3) = r^2/(p^2 + q^2 + r^2) + exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2)\
)^(1/2))*(p^2 + q^2)/(2*p^2 + 2*q^2 + 2*r^2) +
exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 +\
 q^2 + r^2))^(1/2))*(p^2 + q^2)/(2*p^2 + 2*q^2 + 2*r^2),
  (3, 4) = 1/ep*q*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(2*p^2 +\
 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/ep*q*exp(2/ep*i/m\
u*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 + 2*q^2 +
2*r^2)*(ep*mu*p^2 + ep\
*mu*q^2 + ep*mu*r^2)^(1/2),
  (3, 5) = 1/ep*p*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(2*p^2 + \
2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/ep*p*exp(-2/ep*i/m\
u*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^2 + 2*q^2 +
2*r^2)*(ep*mu*p^2 + ep\
*mu*q^2 + ep*mu*r^2)^(1/2),
  (3, 6) = 0,
  (4, 1) = 0,
  (4, 2) = r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 + mu*r\
^2)*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/\
2) - r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 + mu*r^2)*(\
q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2),
  (4, 3) = q*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 + mu*\
r^2)*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1\
/2) - q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 + mu*r^2)*(\
q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2),
  (4, 4) = p^2/(p^2 + q^2 + r^2) + exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2)\
)^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2) +
exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 +\
 q^2 + r^2))^(1/2))*(q^2 + r^2)/(2*p^2 + 2*q^2 + 2*r^2),
  (4, 5) = p*q/(p^2 + q^2 + r^2) - p*q*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2) -
mu*p*q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + \
```

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q^2 + r^2))^(1/2))/(mu*q^2 + mu*r^2)*(q^2 + r^2)/(2*p^2 + 2*q^2 +
2*r^2),
  (4, 6) = p*r/(p^2 + q^2 + r^2) - p*r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(2*p^2 + 2*q^2 + 2*r^2) -
mu*p*r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + \
q^2 + r^2))^(1/2))/(mu*q^2 + mu*r^2)*(q^2 + r^2)/(2*p^2 + 2*q^2 +
2*r^2),
  (5, 1) = 1/2*ep*r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(ep*mu\
*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/2*ep*r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 +\
 q^2 + r^2))^(1/2))/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2),
  (5, 2) = 1/2*ep/p*q*r*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(e\
p*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) -
1/2*ep/p*q*r*exp(2/ep*i/mu*pi*t*(ep*m\
u*(p^2 + q^2 + r^2))^(1/2))/(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2) - r*exp(-\
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(q^3 +
q*r^2)/(mu*q^2 + mu*r^2)\
/(2*p^3 + 2*p*q^2 + 2*p*r^2)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2) + r*exp(\
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(q^3 +
q*r^2)/(mu*q^2 + mu*r^2)\
/(2*p^3 + 2*p*q^2 + 2*p*r^2)*(ep*mu*p^2 + ep*mu*\,
          q^2 + ep*mu*r^2)^(1/2)
  (5, 3) = 1/2*ep/p*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(p^2 + \
q^2)/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) - 1/2*ep/p*exp(-
2/ep*i/mu*pi*t*(\
ep*mu*(p^2 + q^2 + r^2))^(1/2))*(p^2 + q^2)/(ep*mu*p^2 +
ep*mu*q^2 + ep*mu*r^2)\
^(1/2) + q*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(q^3 + q*r^2)/(\
mu*q^2 + mu*r^2)/(2*p^3 + 2*p*q^2 + 2*p*r^2)*(ep*mu*p^2 +
ep*mu*q^2 + ep*mu*r^2\
)^(1/2) - q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))*(q^3 + q*r^2)/(\
mu*q^2 + mu*r^2)/(2*p^3 + 2*p*q^2 + 2*p*r^2)*(ep\,
          *mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2)
  (5, 4) = p*q/(p^2 + q^2 + r^2) + exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2)\
)^(1/2))*(q^3 + q*r^2)/(2*p^3 + 2*p*q^2 + 2*p*r^2) +
exp(2/ep*i/mu*pi*t*(ep*mu*\
(p^2 + q^2 + r^2))^(1/2))*(q^3 + q*r^2)/(2*p^3 + 2*p*q^2 +
2*p*r^2) - 1/2/p*q*e\
xp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2)) -
1/2/p*q*exp(2/ep*i/mu*pi*\
```

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t*(ep*mu*(p^2 + q^2 + r^2))^(1/2)),
  (5, 5) = 1/2*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2)) + 1/2*exp(2\
/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2)) + q^2/(p^2 + q^2 +
r^2) - p*q*ex\
p(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(q^3 +
q*r^2)/(q^2 + r^2)/(2\
*p^3 + 2*p*q^2 + 2*p*r^2) - mu*p*q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2
+ q^2 + r^2))\
^(1/2))*(q^3 + q*r^2)/(mu*q^2 + mu*r^2)/(2*p^3 + 2*p*q^2 +
2*p*r^2),
  (5, 6) = q*r/(p^2 + q^2 + r^2) - p*r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))*(q^3 + q*r^2)/(q^2 + r^2)/(2*p^3 + 2*p*q^2 +
2*p*r^2) - mu*p*r*exp\
(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(q^3 +
q*r^2)/(mu*q^2 + mu*r^2\
)/(2*p^3 + 2*p*q^2 + 2*p*r^2),
  (6, 1) = 1/2*ep*q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(ep*mu*\
p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) - 1/2*ep*q*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 +\
 q^2 + r^2))^(1/2))/(ep*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2),
  (6, 2) = 1/2*ep/p*q^2*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(ep\
*mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2) - 1/2*ep/p*q^2*exp(-
2/ep*i/mu*pi*t*(ep*m\
u*(p^2 + q^2 + r^2))^(1/2))/(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2) + r*exp(-\
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(mu*q^2 +
mu*r^2)/(2*p^3*r + 2*\
p*q^2*r + 2*p*r^3)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2)*(p^2*q^2 + p^2*r^2\
 + q^4 + q^2*r^2) - r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q\
^2 + mu*r^2)/(2*p^3*r + 2*p*q^2*r + 2*p*r^3)*(ep\,
          *mu*p^2 + ep*mu*q^2 + ep*mu*r^2)^(1/2)*(p^2*q^2 +
p^2*r^2 + q^4 + q^\
2*r^2)
  (6, 3) = q*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*q^2 + mu*r\
^2)/(2*p^3*r + 2*p*q^2*r + 2*p*r^3)*(ep*mu*p^2 + ep*mu*q^2 +
ep*mu*r^2)^(1/2)*(\
p^2*q^2 + p^2*r^2 + q^4 + q^2*r^2) - q*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(mu*q^2 + mu*r^2)/(2*p^3*r + 2*p*q^2*r +
2*p*r^3)*(ep*mu*p^2 + ep*\
mu*q^2 + ep*mu*r^2)^(1/2)*(p^2*q^2 + p^2*r^2 + q^4 + q^2*r^2) +
1/2*ep/p*q/r*ex\
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p(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(p^2 +
q^2)/(ep*mu*p^2 + ep*\
mu*q^2 + ep*mu*r^2)^(1/2) - 1/2*ep/p*q/r*exp(2/e\,
           p*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))*(p^2 +
q^2)/(ep*mu*p^2 \
+ ep*mu*q^2 + ep*mu*r^2)^(1/2)
  (6, 4) = p*r/(p^2 + q^2 + r^2) - exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2)\
)^(1/2))/(2*p^3*r + 2*p*q^2*r + 2*p*r^3)*(p^2*q^2 + p^2*r^2 + q^4
+ q^2*r^2) - \
exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(2*p^3*r +
2*p*q^2*r + 2*p*\
r^3)*(p^2*q^2 + p^2*r^2 + q^4 + q^2*r^2) + 1/2/p*q^2/r*exp(-
2/ep*i/mu*pi*t*(ep*\
mu*(p^2 + q^2 + r^2))^(1/2)) +
1/2/p*q^2/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2\
 + r^2))^(1/2)),
  (6, 5) = q*r/(p^2 + q^2 + r^2) - 1/2*q/r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^\
2 + r^2))^(1/2)) - 1/2*q/r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2)) \
+ p*q*exp(-2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(q^2 +
r^2)/(2*p^3*r\
 + 2*p*q^2*r + 2*p*r^3)*(p^2*q^2 + p^2*r^2 + q^4 + q^2*r^2) +
mu*p*q*exp(2/ep*i\
/mu*pi*t*(ep*mu*(p^2 + q^2 + r^2))^(1/2))/(mu*q^2 +
mu*r^2)/(2*p^3*r + 2*p*q^2*\
r + 2*p*r^3)*(p^2*q^2 + p^2*r^2 + q^4 + q^2*r^2),
  (6, 6) = r^2/(p^2 + q^2 + r^2) + p*r*exp(-
2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 + \
r^2))^(1/2))/(q^2 + r^2)/(2*p^3*r + 2*p*q^2*r + 2*p*r^3)*(p^2*q^2
+ p^2*r^2 + q\
^4 + q^2*r^2) + mu*p*r*exp(2/ep*i/mu*pi*t*(ep*mu*(p^2 + q^2 +
r^2))^(1/2))/(mu*\
q^2 + mu*r^2)/(2*p^3*r + 2*p*q^2*r + 2*p*r^3)*(p^2*q^2 + p^2*r^2
+ q^4 + q^2*r^\
2)
)
```

After some algebra, the kernel matrix $\mathbf{H}(p,q,r,t)$ simplifies to

$$\mathbf{H}(p,q,r,t)=$$

$$\begin{bmatrix} 1-(q^2+r^2)F & pqF & prF & 0 & -ir\eta G & iq\eta G \\ pqF & 1-(p^2+r^2)F & qrF & ir\eta G & 0 & -ip\eta G \\ prF & qrF & 1-(p^2+q^2)F & -iq\eta G & ip\eta G & 0 \\ 0 & irG/\eta & -iqG/\eta & 1-(q^2+r^2)F & pqF & prF \\ -irG/\eta & 0 & ipG/\eta & pqF & 1-(p^2+r^2)F & qrF \\ iqG/\eta & -ipG/\eta & 0 & prF & qrF & 1-(p^2+q^2)F \end{bmatrix},$$

(B.14)

where $F=(1-\cos(2\pi k\upsilon t))/k^2$, $G=\sin(2\pi k\upsilon t)/k$, $k=\sqrt{p^2+q^2+r^2}$, $\upsilon=1/\sqrt{\varepsilon\mu}$, and $\eta=\sqrt{\mu/\varepsilon}$.